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Some thoughts on phase-space representations in gyrotron theory

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The paper is divided into two main parts. The first gives a formal representation of perturbation theory in terms of manifolds in phase space, with particular reference to gyrotron theory. The second considers the extension of the linear coupled modes theory for the gyrotron travelling-wave tube presented in Lindsay *et al.* (1982) to include an electron density distribution. We call the resulting theory a 'hybrid' model. The paper concludes with the development of a dispersion equation for this model using a full Fourier–Laplace transform method.

1. Introduction

Something of a revolution has been occurring since the mid 1950s in the mathematical presentation of physics. Central to this new approach are the dual vector spaces of tangent vectors (derivative operators) $\{\partial/\partial x^i\}$ and differentials $\{dx^i\}$ on manifolds. In this paper, we use this theory only in an elementary way, so we will not give a detailed account of it: many good references for it are now available (Nickerson *et al.* 1959, Flanders 1963, Bishop and Goldberg 1968, Abraham and Marsden 1978, Deschamps 1981). So far this approach has not been widely used in plasma physics nor in microwave engineering. We introduce it here to construct a formal mathematical framework in which to classify rigorously the various mathematical models used in the theory of the gyrotron. In particular, we can thereby give more formal definitions of such concepts as filamentary beams and polarization variables than the conventional ones typified by, for example, the series of papers by Haus and Bobroff (see, for example, Haus and Bobroff (1957)).

In the main part of the paper, we then proceed to set up some sort of bridge between the elementary models and the Vlasov equation analyses by developing what we will call a 'hybrid' method. We retain a relatively rigorous and general approach throughout which, we believe, gives an unexpectedly powerful physical insight into the theory. In this paper, we conclude with the development of a typical dispersion equation for a gyro-TWT. This parallels the presentation in Lindsay *et al.* (1982).

2. Phase-space representations

An electron trajectory is simply a space curve in phase space. Through every point in phase space passes one unique trajectory, in a specific direction. If the phase space is n-dimensional, we can specify this direction by a single tangent vector or by $(n-1)$ relations between differentials. The actual trajectory is an *integral curve* of the tangent vector.

For simplicity, consider the four-space parametrized by the coordinates x, y, z, and t. A trajectory in this 'real' space is a curve λ, which is a function from some

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parameter u into the x, y, z, t space, or

$$\lambda : u \rightarrow x, y, z, t$$

In terms of differentials, the direction of λ at any point can be specified by the three relations:

$$\left.\begin{aligned} \lambda^* \, dz &= v_z(x, y, z, t)\lambda^* \, dt \\ \lambda^* \, dy &= v_y(x, y, z, t)\lambda^* \, dt \\ \lambda^* \, dx &= v_x(x, y, z, t)\lambda^* \, dt \end{aligned}\right\} \tag{2.1}$$

for given functions v_x, v_y, v_z, where the λ^* notation indicates that, for example, the integral of dz along λ equals that of $v_z \, dt$ along λ for an arbitrary interval in u, or

$$\int_{u_1}^{u_2} \lambda^* \, dz = \int_{u_1}^{u_2} \frac{d\lambda_z}{du} \, du = \int_{u_1}^{u_2} v_z \frac{d\lambda_t}{du} \, du = \int_{u_1}^{u_2} v_z \lambda^* \, dt$$

where λ_z is the z component of the λ function: $z = \lambda_z(u)$ along λ, and likewise λ_t.

The corresponding tangent vector is simply the directional derivative operator along λ, or d/du. Now this can be written as

$$\frac{d}{du} = \frac{dx}{du} \partial_x + \frac{dy}{du} \partial_y + \frac{dz}{du} \partial_z + \frac{dt}{du} \partial_t \tag{2.2}$$

using the notation ∂_x for $\partial/\partial x$, and from the equations in differentials, since the total derivatives along λ appear in

$$\lambda^* \, d\xi = \frac{d\xi}{du} \, du$$

for an arbitrary variable ξ, we can express this as

$$\frac{d}{du} = \frac{dt}{du} [v_x(\mathbf{x}, t)\partial_x + v_y(\mathbf{x}, t)\partial_y + v_z(\mathbf{x}, t)\partial_z + 1\partial_t] \tag{2.3}$$

writing $(\mathbf{x}, t)$ for (x, y, z, t). Note how ∂_x, ∂_y, ∂_z and ∂_t form a vector basis for all directional derivatives at an arbitrary point of the four-space.

Turning to the seven-dimensional phase-space with coordinates $x, y, z, v_x, v_y, v_z, t$, which we call the space N, let us consider a cycloidal motion modified by a transverse electric field as described by the equations

$$\left.\begin{aligned} \tau^* \, dz &= v_{z1}\tau^* \, dt \\ \tau^* \, dy &= v_y(\mathbf{x}, \mathbf{v}, t)\tau^* \, dt \\ \tau^* \, dx &= v_x(\mathbf{x}, \mathbf{v}, t)\tau^* \, dt \\ \tau^* \, dv_z &= 0 \\ \tau^* \, dv_y - \frac{e}{m} B_0 v_x(\mathbf{x}, \mathbf{v}, t)\tau^* \, dt &= -\frac{e}{m} E_y \tau^* \, dt \\ \tau^* \, dv_x + \frac{e}{m} B_0 v_y(\mathbf{x}, \mathbf{v}, t)\tau^* \, dt &= -\frac{e}{m} E_x \tau^* \, dt \end{aligned}\right\} \tag{2.4}$$

representing the trajectory by τ, and assuming a constant z-velocity $v_z = v_{z1}$. We will also use τ to represent the *parameter* of the trajectory τ, since the two never

occupy an ambiguous position in our notation. Now in N, a basis for tangent vectors is given by the set

$$\partial_x, \partial_y, \partial_z, \partial_{v_x}, \partial_{v_y}, \partial_{v_z}, \partial_t$$

So the directional derivative along τ must admit a representation

$$\frac{d}{d\tau} = \sum \xi^i \partial_{x^i} + \sum \psi^i \partial_{v^i} + \sigma \partial_t$$

for some functions $\xi^1, \xi^2, \ldots, \psi^3$ and σ, and representing x by x^1, y by x^2, etc. Now by the chain rule, it must hold that

$$\frac{dx^i}{d\tau} = \xi^i, \quad \frac{dv^i}{d\tau} = \psi^i, \quad \frac{dt}{d\tau} = \sigma$$

so from our equations in differentials we can write

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\left[v_x \partial_x + v_y \partial_y + v_z \partial_z - \frac{e}{m}[B_0 v_y + E_x]\partial_{v_x} + \frac{e}{m}[B_0 v_x - E_y]\partial_{v_y} + 0\partial_{v_z} + \partial_t \right] \quad (2.5)$$

where the ∂_{v_z} term just drops out.

If we choose the parameter τ to be uniform with t so that $dt/d\tau = 1$, this expression takes the form given below, which we refer to as the *Vlasov operator* for this motion:

$$\mathrm{VL} = v_x \partial_x + v_y \partial_y + v_{z1} \partial_z - \frac{e}{m}[B_0 v_y + E_x]\partial_{v_x} + \frac{e}{m}[B_0 v_x - E_y]\partial_{v_y} + \partial_t \quad (2.6)$$

We define a *datum* m-space as an arbitrary m-dimensional subspace of phase space which intersects with any arbitrary trajectory at most once. Thus our trajectories will 'draw out' a datum m-space into an $(m+1)$-space and m can be at most 6 in our space N. We choose the term *datum* because our trajectories are *characteristics* for the PDE $\mathrm{VL}(f) = 0$ so a *datum* space is one on which 'initial' data for any solution f of this PDE can be given.

A *filamentary beam* can then be defined as the two-space generated in phase space from an arbitrary one-dimensional datum space; i.e. 'initial' data for a filamentary beam is given in terms of a *space curve* or *filament* in phase space (say, in the hyperplanes $z = 0$ or $t = 0$). The resulting two-space—an ordinary surface—in phase space we will call $\mathscr{F}$. Now, because $\mathscr{F}$ is only a two-dimensional manifold, we only need two coordinates to parametrize it. We choose z and t. Because $\mathscr{F}$ is, by definition, generated from trajectories from the initial datum one-space, at any point of $\mathscr{F} \cdot d/d\tau$ must lie in (i.e. be tangent to) $\mathscr{F}$. Any tangent vector of $\mathscr{F}$ can be expressed in terms of the basis of just the two vectors:

$$\partial_z\Big|_{\mathscr{F}} \quad \text{and} \quad \partial_t\Big|_{\mathscr{F}}$$

Where these ∂_z and ∂_t are *not the same* as those of the surrounding phase space: for example $\partial_z|_{\mathscr{F}}$ is taken in the direction of a vector lying in or tangent to $\mathscr{F}$ in a constant-t plane.

The only relationship in our τ^*-equations still surviving as a relationship between coordinates of $\mathscr{F}$ is $\tau^*\, dz = v_{z1}\tau^*\, dt$, so we have

$$\frac{d}{d\tau} = \frac{dz}{d\tau}\,\partial_z\Big|_{\mathscr{F}} + \frac{dt}{d\tau}\,\partial_t\Big|_{\mathscr{F}}$$

$$= \frac{dt}{d\tau}\,(v_{z1}\partial_z + \partial_t) = v_{z1}\partial_z + \partial_t \tag{2.7}$$

for τ chosen uniform with t, and writing $\partial_z|_{\mathscr{F}}$, $\partial_t|_{\mathscr{F}}$ as just ∂_z, ∂_t. Since for any ξ, $\tau^*\, d\xi = (d\xi/d\tau)\, d\tau$ we can rewrite the remaining τ^*-equations as

$$\left.\begin{aligned} &\frac{d}{d\tau}\,x = v_x, \quad \frac{d}{d\tau}\,y = v_y \\ &\frac{d}{d\tau}\,v_x = -\frac{e}{m}\,[B_0 v_y + E_x] \\ &\frac{d}{d\tau}\,v_y = \frac{e}{m}\,[B_0 v_x - E_y] \end{aligned}\right\} \tag{2.8}$$

if $dt/d\tau = 1$, but, within $\mathscr{F}$, $d/d\tau = v_{z1}\partial_z + \partial_t$ so we obtain the equations

$$\left.\begin{aligned} &v_{z1}\partial_z x + \partial_t x = v_x \\ &v_{z1}\partial_z y + \partial_t y = v_y \\ &v_{z1}\partial_z v_x + \partial_t v_x = -\frac{e}{m}\,[B_0 v_y + E_x] \\ &v_{z1}\partial_z v_y + \partial_t v_y = \frac{e}{m}\,[B_0 v_x - E_y] \end{aligned}\right\} \tag{2.9}$$

which give $x(z, t)$, $y(z, t)$, $v_x(z, t)$, $v_y(z, t)$ as functions *over the surface* $\mathscr{F}$.

A *sheet beam* is a three-space generated from a datum two-space. We now need three variables to parametrize this space $\mathscr{S}$. We might choose, x, z, t; $d/d\tau$ will now have the expansion

$$\frac{d}{d\tau} = \frac{dx}{d\tau}\,\partial_x + \frac{dz}{d\tau}\,\partial_z + \frac{dt}{d\tau}\,\partial_t$$

But an alternative is to choose a cycloidal angle defined by an $(x, y) \to (r, \theta)$ or $(v_x, v_y) \to (v_\perp, \phi)$ transformation. If the **E** field in our motion is small, then we can say that

$$\frac{d}{d\tau} = \frac{d\phi}{d\tau}\,\partial_\phi + \frac{dz}{d\tau}\,\partial_z + \frac{dt}{d\tau} \approx \omega_{ce1}\partial_\phi + v_{z1}\partial_z + \partial_t \tag{2.10}$$

for ω_{ce1} equal to $(e/m)B_0$ at the datum space, assumed a common constant. Using this parametrization will preserve a linear form for our equations overall, at least as a first-order approximation. Spatial and velocity cycloidal angles are often used interchangeably: this strictly requires that the sheet beam have the form of an *annulus* (as Lau (1982)): these also correspond to the 'beamlet' models (as, for example, Döhler and Friz (1982)).

If we use another four variables which are *themselves* constants of the motion,

such as the initial position and velocity at some *entry plane* $z = z_1$ (a datum space for our cycloidal motion) or $x_1, y_1, v_{\perp 1}, v_{z1}$, we can reparametrize the whole phase-space N in terms of these variables by the transformation:

$$x = x(x_1, y_1, v_{\perp 1}, v_{z1}, \phi, z, t)$$

$$y = y(x_1, y_1, v_{\perp 1}, v_{z1}, \phi, z, t)$$

.....

.....

$$v_y = v_y(x_1, y_1, v_{\perp 1}, v_{z1}, \phi, z, t)$$

$$\phi = \phi, \quad z = z, \quad t = t$$

which is tantamount to solving the equations of motion. Note that because

$$\frac{dx_1}{d\tau} = \frac{dy_1}{d\tau} = \frac{dv_{\perp 1}}{d\tau} = \frac{dv_{z1}}{d\tau} = 0$$

$d/d\tau$ is still of the form $\omega_{ce1}\partial_\phi + v_{z1}\partial_z + \partial_t$ given above.

3. Phase spaces with perturbation

We can set up analogous models for perturbation theories by defining a 13-space P with coordinates which we label $x^{(0)}, x^{(\bar{0})}, y^{(0)}, y^{(\bar{0})}, z^{(0)}, z^{(\bar{0})}, v_x^{(0)}, v_x^{(\bar{0})}, v_y^{(0)}, v_y^{(\bar{0})}, v_z^{(0)}, v_z^{(\bar{0})}, t$. The barred ($^{(\bar{0})}$) coordinates we will call *zero-order complement* rather than first order: subsequently we approximate these by variables $x^{(1)}, y^{(1)}$, etc., obeying linearized equations. We further define the trivial mapping from P to N or $\mathcal{N}$: $P \rightarrow N$ by $\mathcal{N}(\mathbf{x}^{(0)}, \mathbf{x}^{(\bar{0})}, t) = (\mathbf{x}^{(0)} + \mathbf{x}^{(\bar{0})}, t)$ writing $\mathbf{x}^{(0)}$ for $x^{(0)}, y^{(0)}, z^{(0)}, v_x^{(0)}, v_y^{(0)}, v_z^{(0)}$, likewise $\mathbf{x}^{(\bar{0})}$; i.e. $\mathcal{N}$ identifies that point of N where $x = x^{(0)} + x^{(\bar{0})}$, $y = y^{(0)} + y^{(\bar{0})}$, $\ldots, v_z = v_z^{(0)} + v_z^{(\bar{0})}, t = t$ with the point $x^{(0)}, x^{(\bar{0})}, y^{(0)}, y^{(\bar{0})}, \ldots, t$ of P.

A datum space D^P in P corresponding to a given datum space D^N in N is just the unique subspace of P lying in the $\mathbf{x}^{(\bar{0})} = 0$ or $x^{(\bar{0})} = y^{(\bar{0})} = \ldots = v_z^{(\bar{0})} = 0$ hyperplane such that $\mathcal{N}(D^P) = D^N$. Thus if $(\mathbf{x}, t)$ is a point in D^N, $(\mathbf{x}, \mathbf{0}, t)$ is a point in D^P! This can then be, at most, a six-dimensional subspace of P.

Now from any datum space in P we define two manifolds drawn out by the trajectories: one obeys a specially selected set of *zero-order equations* and lies entirely in the $\mathbf{x}^{(\bar{0})} = 0$ or $x^{(\bar{0})} = y^{(\bar{0})} = \ldots = v_z^{(\bar{0})} = 0$ hyperplane. We call this the *zero-order solution* (space) and denote this manifold by $\mathcal{T}^{(0)}$ or $\mathcal{T}_D^{(0)}$ if the datum space is D (in N or P). Evidently, because $\mathbf{x}^{(\bar{0})} = 0$ here, the equations in $x^{(0)}, y^{(0)}, \ldots, v_z^{(0)}, t$ describing $\mathcal{T}^{(0)}$ will be the same as those in $x, y, \ldots, v_z, t$ describing $\mathcal{N}(\mathcal{T}^{(0)})$ in N.

The second manifold is that obeying the *actual equations* in that, denoting the manifold by $\mathcal{T}$, $\mathcal{N}(\mathcal{T})$ is the solution space in N of our actual dynamical equations. $\mathcal{T}$ is made unique by the requirement that the zero-order variables on $\mathcal{T}$ obey the zero-order equations, or the *projection* of $\mathcal{T}$ onto the $\mathbf{x}^{(\bar{0})} = 0$ hyperplane is just $\mathcal{T}^{(0)}$.

Under this model, $\mathcal{T}$ and $\mathcal{T}^{(0)}$ can at most be seven-dimensional manifolds in P, and so are analogous to $\mathcal{F}$ and $\mathcal{S}$ in N, in that they represent hypersurfaces in P parametrizable by a restricted set of variables. Furthermore, if we take the projection of $\mathcal{T}$ onto some $\mathbf{x}^{(0)} = \mathbf{k}$ hyperplane; i.e. look at just the $x^{(\bar{0})}, y^{(\bar{0})}, \ldots, v_z^{(\bar{0})}, t$

variation, *here* all trajectories emanate from the origin because of the definition of the datum space in P. Note, incidentally, how *trajectories* are subsumed under these definitions as the $\mathscr{T}$ spaces deriving from a *point* datum space.

Now once our datum space is six-dimensional, then *every* point of N is a member of the solution space. But $\mathscr{T}$ and $\mathscr{T}^{(0)}$ here are still just seven-dimensional hypersurfaces in P, with the property now that $\mathscr{N}(\mathscr{T}) = N$, $\mathscr{N}(\mathscr{T}^{(0)}) = N$. So we can speak of the notion of *corresponding points* of $\mathscr{T}$ and $\mathscr{T}^{(0)}$ as mapping under $\mathscr{N}$ into *the same* point of N: thus if $(\mathbf{x}^{(0)}, \mathbf{x}^{(\bar{0})}, t)$ is in $\mathscr{T}$ and $(\mathbf{x}'^{(0)}, \mathbf{0}, t)$ is in $\mathscr{T}^{(0)}$ then these are corresponding points if $\mathbf{x}^{(0)} + \mathbf{x}^{(\bar{0})} = \mathbf{x}'^{(0)}$. Using this notion, we can define, given a density function n over the datum space and knowing $dn/d\tau = 0$ so that n is known over the actual space $\mathscr{T}$ and over $\mathscr{T}^{(0)}$, a *zero-order density function* $n^{(0)}$ over $\mathscr{T}$ by

$$n^{(0)}(\mathbf{x}^{(0)}, \mathbf{x}^{(\bar{0})}, t) = n(\mathbf{x}'^{(0)}, \mathbf{0}, t) \tag{3.1}$$

where $(\mathbf{x}^{(0)}, \mathbf{x}^{(\bar{0})}, t)$ is in $\mathscr{T}$ and $(\mathbf{x}'^{(0)}, \mathbf{0}, t)$ is the corresponding point of $\mathscr{T}^{(0)}$. This agrees with the usual notion that this is the zero-order density at 'the same' point. Observe that there will now also be a trajectory in $\mathscr{T}$ itself containing a point $(\mathbf{x}'^{(0)}, \mathbf{x}'^{(\bar{0})}, t)$ where $\mathbf{x}'^{(0)}$ equals that on the $\mathscr{T}^{(0)}$ point above. So $n^{(0)}$ at any point in $\mathscr{T}$ is in fact equal to the 'actual' density n at some nearby point in $\mathscr{T}$, because $n(\mathbf{x}^{(0)}, \mathbf{x}^{(\bar{0})}, t)$ in $\mathscr{T}$ will equal $n(\mathbf{x}^{(0)}, \mathbf{0}, t)$ in $\mathscr{T}^{(0)}$ since both emanate from the same datum space point.

Our equations for $\mathscr{T}$ will be of the form:

$$\frac{d}{d\tau}\mathbf{x}^{(0)} = \boldsymbol{\psi}^{(0)}, \quad \frac{d}{d\tau}\mathbf{x}^{(\bar{0})} = \boldsymbol{\psi}^{(\bar{0})}$$

and for $\mathscr{T}^{(0)}$:

$$\frac{d}{d\tau}\mathbf{x}^{(0)} = \boldsymbol{\psi}^{(0)}, \quad \mathbf{x}^{(\bar{0})} = \mathbf{0}$$

In general, we approximate further, linearizing the $\mathbf{x}^{(\bar{0})}$, $\boldsymbol{\psi}^{(\bar{0})}$ equations to give us equations

$$\frac{d}{d\tau}\mathbf{x}^{(0)} = \boldsymbol{\psi}^{(0)}, \quad \frac{d}{d\tau}\mathbf{x}^{(1)} = \boldsymbol{\psi}^{(1)}$$

where $\mathbf{x}^{(1)}$, $\boldsymbol{\psi}^{(1)}$ approximate $\mathbf{x}^{(\bar{0})}$, $\boldsymbol{\psi}^{(\bar{0})}$. We can then define $\mathbf{x}^{(\bar{1})}$, $\boldsymbol{\psi}^{(\bar{1})}$ such that $\mathbf{x}^{(\bar{0})} = \mathbf{x}^{(1)} + \mathbf{x}^{(\bar{1})}$ and hence define a 19-space Q for the second-order perturbation with $\mathbf{x}^{(0)}$, $\mathbf{x}^{(1)}$, $\mathbf{x}^{(\bar{1})}$, t and a mapping $\mathscr{Q}$: $Q \to N$ with $\mathscr{Q}(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(\bar{1})}, t) = (\mathbf{x}^{(0)} + \mathbf{x}^{(1)} + \mathbf{x}^{(\bar{1})}, t)$, and manifolds $\mathscr{T}$ (actual), $\mathscr{T}^{(0)}$ $(\mathbf{x}^{(1)} = \mathbf{x}^{(\bar{1})} = 0)$, $\mathscr{T}^{(1)}$ $(\mathbf{x}^{(\bar{1})} = 0)$, etc. Note that $n^{(0)}$, such that $n = n^{(\bar{0})} + n^{(0)}$ on $\mathscr{T}$, $n^{(1)}$, $n^{(\bar{1})}$ etc., can all be defined analogously.

The Vlasov equation needs some care. We have

$$\frac{d}{d\tau}(n) = \frac{d}{d\tau}(n^{(0)} + n^{(\bar{0})}) = \frac{d}{d\tau}n^{(0)} + \frac{d}{d\tau}n^{(\bar{0})} = 0 \tag{3.2}$$

on $\mathscr{T}$. But we cannot presume now that

$$\frac{d}{d\tau}n^{(0)} = 0$$

For the $d/d\tau$ direction on $\mathscr{T}$ does not necessarily correspond to that on $\mathscr{T}^{(0)}$. Suppose τ is a trajectory in $\mathscr{T}$. Then $n^{(0)}$ along τ equals n along the τ' curve in $\mathscr{T}^{(0)}$

described by the corresponding points of τ. But τ' *may not correspond to a trajectory* in $\mathscr{T}^{(0)}$—it may be cutting across trajectories. So we cannot presume it to be constant along τ' and so $n^{(0)}$ constant along τ.

The solution in fact comes out quite naturally. We need 7 variables to parametrize $\mathscr{T}$ in P, so we choose the *actual variables* $\mathbf{x} = \mathbf{x}^{(0)} + \mathbf{x}^{(\bar{0})}$ of N, which of course are the zero-order $\mathbf{x}^{(0)}$ variables of the *corresponding points* of $\mathscr{T}^{(0)}$. Putting $\mathbf{x}$ as $\{x^i\}$, we must have

$$\frac{d}{d\tau} = \sum_{i=1}^{6} \frac{dx^i}{d\tau}\,\partial_{x^i} + \partial_t = \sum_{i=1}^{6} \phi^i(\mathbf{x}, t)\partial_{x^i} + \partial_t$$

but in $\mathscr{T}^{(0)}$ we have similarly, using the $\mathbf{x}^{(0)}$ variables to parametrize *this*

$$\frac{d}{d\tau} = \sum_{i=1}^{6} \frac{dx^{i(0)}}{d\tau}\,\partial_{x^{i(0)}} + \partial_t = \sum_{i=1}^{6} \phi^{i(0)}(\mathbf{x}^{(0)}, t)\partial_{x^{i(0)}} + \partial_t$$

But these are *the same variables now*: we are in effect parametrizing *both* $\mathscr{T}$ and $\mathscr{T}^{(0)}$ by the zero-order variables of the corresponding points in $\mathscr{T}^{(0)}$.

Applying this to a (classical) system in which the zero-order motion is cycloidal, with an imposed RF excitation contributed by $\mathbf{E}$, $\mathbf{B}$ fields, and defining $\omega_{ce0} = (e/m)B_0$, then since

$$\left.\begin{aligned} &\frac{d}{d\tau}x = v_x, \quad \frac{d}{d\tau}y = v_y, \quad \frac{d}{d\tau}z = v_z \\ &\frac{d}{d\tau}v_x = -\left(\omega_{ce0}\,v_y + \frac{e}{m}\,[E_x + v_y B_z - v_z B_y]\right) \\ &\quad\vdots \end{aligned}\right\} \tag{3.3}$$

we obtain on $\mathscr{T}$:

$$\begin{aligned} \frac{d}{d\tau}n^{(\bar{0})} + \Bigg[& v_x\partial_x n^{(0)} + v_y\partial_y n^{(0)} + v_z\partial_z n^{(0)} \\ & - \left(\omega_{ce0}\,v_y + \frac{e}{m}\,[E_x + v_y B_z - v_z B_y]\right)\partial_{v_x} n^{(0)} \\ & + \left(\omega_{ce0}\,v_x - \frac{e}{m}\,[E_y + v_z B_x - v_x B_z]\right)\partial_{v_y} n^{(0)} \\ & - \frac{e}{m}\,[E_z + v_x B_y - v_y B_x]\partial_{v_z} n^{(0)} + \partial_t n^{(0)}\Bigg] = 0 \end{aligned} \tag{3.4}$$

But $x = x'^{(0)}$, $v_x = v_x'^{(0)}$, etc., of the corresponding point of $\mathscr{T}^{(0)}$ and the present $n^{(0)}(\mathbf{x}^{(0)}, \mathbf{x}^{(\bar{0})}, t)$ is equal to $n(\mathbf{x}'^{(0)}, \mathbf{0}, t)$ there, so we can pull out

$$v_x\partial_x n^{(0)} + v_y\partial_y n^{(0)} + v_z\partial_z n^{(0)} - \omega_{ce0}\,v_y\partial_{v_x} n^{(0)} + \omega_{ce0}\,v_x\partial_{v_y} n^{(0)} + \partial_t n^{(0)} = 0$$

to obtain

$$\begin{aligned} \frac{d}{d\tau}n^{(\bar{0})} = & -\frac{e}{m}\,[E_x + v_y B_z - v_z B_y]\partial_{v_x} n^{(0)} - \frac{e}{m}\,[E_y + v_z B_x - v_x B_z]\partial_{v_y} n^{(0)} \\ & - \frac{e}{m}\,[E_z + v_x B_y - v_y B_x]\partial_{v_z} n^{(0)} \end{aligned} \tag{3.5}$$

which is the normal 'first-order' Vlasov equation.

In a relativistic system, Liouville's theorem fails in $\mathbf{x}$, $\mathbf{v}$, t space, so we replace $\mathbf{v}$ by $\mathbf{u} = \gamma\mathbf{v}$ where $\gamma = [1 - (v^2/c^2)]^{-1/2}$, as in Baldwin *et al.* (1969).

4. A classification of gyrotron theories

In terms of N- and P-spaces, we can easily see the main division of gyrotron theory. Currently thought to be the most general and comprehensive are the theories involving an explicit density function n where the dynamical equations 'disappear' into the Vlasov equation. We will call these *Vlasov theories.* Most of the elementary treatments, however, work without a density function, and consider the conformation of a filamentary or sheet beam. We will call these *beam conformation* theories. In the present context, the former involve n given on some six-dimensional subspace of N, so that the datum space *fills* some hyperplane of N, whilst the latter have a *filament* or *sheet datum* space, with the electron density regarded (in effect) as constant thereon, and look at the *shape* of the resulting space generated by the trajectories. In these *beam conformation* theories, the bunching that is integral to the gyrotron's amplification mechanism must appear as some sort of corrugation of an 'initially' regular datum space.

Now the trajectories are in fact the same in both cases and so the final information derived really ought to be the same. In particular, there is a strong intuitive feeling that we can somehow 'add up' *filamentary* or *sheet* beams to recover a 'filled space' theory from the beam conformation theory without the direct intervention of the Vlasov equation. There are good reasons for trying to do this: the beam conformation theories give a much richer intuitive picture of what is actually going on. In particular, the role of relativity is brought out markedly more clearly: in the Vlasov methods, this is hidden in the relativistic velocity variable $\mathbf{u} = \gamma\mathbf{v}$ necessary to recover Liouville's theorem. This is why all elementary presentations of the gyrotron mechanism (such as Lau (1982), Döhler and Friz (1982), Sprangle and Drobot (1977), or Mourier (1980)) work from *beam conformation* models. The rest of this paper, therefore, attempts to set up a rigorous extension of beam conformation theories to include a density function. We call the resulting formulation a *hybrid* theory.

5. Background to a hybrid theory

We develop the theory in terms of a configuration representing a simplified gyro-TWT. This we take to be a semi-infinite cylinder of constant cross-section, which we hereafter call the 'waveguide', open-ended to the right ($z \to +\infty$); we do not presume any *particular* cross-section. At some $z = z_1$, we define an *entry plane* and, presuming $v_z > 0$ for all electrons at all times, this becomes a valid datum space in the sense of our original definition. We imagine an electron gun somewhere to the left of the entry plane which serves to set up a known electron distribution n_1 at the entry plane, and that at, or immediately behind, the entry plane an *input waveguide* injects an EM wave (the 'source') which will interact with the beam to perturb the beam distribution increasingly with increasing z. The zero-order solution is taken to correspond to that for no source wave.

In extending the theory to a *hybrid* formulation, the equations of motion give us no new problems. We reparametrize the whole of the phase space N, in effect, in terms of annular beamlets by choosing the variables, x_1, y_1, $v_{\perp 1}$, v_{z1}, ϕ, z and t. So each $(x_1, y_1, v_{\perp 1}, v_{z1})$ defines a kind of 'virtual' annular beamlet which will be para-

metrized by ϕ, z and t. $v_\perp$ is defined by $v_x = v_\perp \sin\phi$, $v_y = v_\perp \cos\phi$. We leave ϕ as a 'current' parameter rather than use the initial ϕ_1 because the ϕ-variation will give us precisely the corrugation or bunching that we are looking for. The relativistic formulation gives us four equations of motion:

$$\left.\begin{aligned}
\frac{d}{d\tau}(\gamma v_x) + \frac{eB_0}{m_0} v_y &= -\frac{e}{m_0}[E_x + v_y B_z - v_z B_y] \\
\frac{d}{d\tau}(\gamma v_y) - \frac{eB_0}{m_0} v_x &= -\frac{e}{m_0}[E_y + v_z B_x - v_x B_z] \\
\frac{d}{d\tau}(\gamma v_z) &= -\frac{e}{m_0}[E_z + v_x B_y - v_y B_x] \\
\frac{d}{d\tau}(\gamma) &= -\frac{e}{m_0 c^2}\,\mathbf{E}\cdot\mathbf{v}
\end{aligned}\right\} \tag{5.1}$$

where $\gamma = [1 - (v^2/c^2)]^{-1/2}$ as usual.

Again, by expanding into the space P, we assume simple cycloidal motion for the zero-order behaviour, which corresponds to setting all the RHSs above equal to zero. Hence $\gamma^{(0)} = \gamma_1$—the constant initial value—and, taking the first equation as an example, we obtain on the LHS:

$$-\frac{e}{m_0 c^2}\sum_i E_i(v_i^{(0)} + v_i^{(\bar{0})})(v_x^{(0)} + v_x^{(\bar{0})}) + (\gamma_1 + \gamma^{(\bar{0})})\frac{d}{d\tau}(v_x^{(0)} + v_x^{(\bar{0})}) + \frac{eB_0}{m_0}(v_y^{(0)} + v_y^{(\bar{0})})$$

using the γ-equation, and writing v_i for v_x or v_y or v_z; on the RHS:

$$-\frac{e}{m_0}[E_x + (v_y^{(0)} + v_y^{(\bar{0})})B_z - (v_z^{(0)} + v_z^{(\bar{0})})B_y]$$

Taking the E_i and B_i (E_x, E_y, E_z, B_x, B_y, B_z) field components to be first-order, we obtain on linearizing (retaining only first-order products) and subtracting the zero-order equation $dv_x^{(0)}/d\tau = -(eB_0/m_0\gamma_1)v_y^{(0)}$:

$$\frac{d}{d\tau}v_x^{(1)} + \omega_{ce1}v_y^{(1)} - \gamma^{(1)}\frac{\omega_{ce1}}{\gamma_1}v_y^{(0)} = \frac{e}{m_0\gamma_1 c^2}\sum_i E_i v_i^{(0)} v_x^{(0)} - \frac{e}{m_0\gamma_1}[E_x + v_y^{(0)}B_z - v_z^{(0)}B_y] \tag{5.2}$$

defining $\omega_{ce1} = eB_0/m_0\gamma_1 = \omega_{ce0}/\gamma_1$. Remember also $v_z^{(0)} = v_{z1}$, $dv_z^{(0)}/d\tau = 0$. The remaining equations give:

$$\frac{d}{d\tau}v_y^{(1)} - \omega_{ce1}v_x^{(1)} + \frac{\omega_{ce1}}{\gamma_1}v_x^{(0)}\gamma^{(1)} = \frac{e}{m_0\gamma_1 c^2}\sum_i E_i v_i^{(0)} v_y^{(0)} - \frac{e}{m_0\gamma_1}[E_y + v_{z1}B_x - v_x^{(0)}B_z] \tag{5.3}$$

$$\frac{d}{d\tau}v_z^{(1)} = \frac{e}{m_0\gamma_1 c^2}\sum_i E_i v_i^{(0)} v_{z1} - \frac{e}{m_0\gamma_1}[E_z + v_x^{(0)}B_y - v_y^{(0)}B_x] \tag{5.4}$$

$$\frac{d}{d\tau}\gamma^{(1)} = \frac{d}{d\tau}\gamma = -\frac{e}{m_0 c^2}\sum_i E_i v_i^{(0)} \tag{5.5}$$

this last now becoming an equation integral to our set. Note here that these equations now give v_x, v_y, v_z, and γ as functions of x_1, y_1, $v_{\perp 1}$, v_{z1}, ϕ, z and t, but that only the ϕ, z and t dependence is here determined because x_1, y_1, $v_{\perp 1}$, v_{z1} (the entry-plane values of x, y, $v_\perp$ and v_z) are constants of the motion.

The dynamical equations are completed by:

$$\frac{dx^{(1)}}{d\tau} = v_x^{(1)}, \quad \frac{dy^{(1)}}{d\tau} = v_y^{(1)}, \quad \frac{dz^{(1)}}{d\tau} = v_z^{(1)} \tag{5.6}$$

now giving the first-order $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ dependence on ϕ, z and t. We will in fact choose $\phi^{(0)}$ and $z^{(0)}$ to parametrize our solutions rather than the actual ϕ and z subsequently. The distinction does not of course apply to x_1, y_1, $v_{\perp 1}$, v_{z1} which relate to the datum space.

We choose EM equations that will again give us explicit z and t dependence by using the separation defined in Marcuvitz and Schwinger (1951), which we also used in Lindsay *et al.* (1982). Now in the normal development of these equations, exp ($j\omega t$) dependence of all variables is assumed from the start. Partly to be consistent and partly because of the much enhanced physical insight obtained, we will present these in their general form instead—for arbitrary z, t dependence *to be determined from our total self-consistent set of equations*—and take a Fourier–Laplace transform of all our equations together.

So, defining

$$\left.\begin{aligned} \mathbf{E} &= \sum_n V_n(z, t)\mathbf{e}_{\perp n}(x, y) + \sum_n q_n(z, t)\mathbf{e}_{zn}(x, y) \\ \mathbf{H} &= \sum_n I_n(z, t)\mathbf{h}_{\perp n}(x, y) + \sum_n p_n(z, t)\mathbf{h}_{zn}(x, y) \end{aligned}\right\} \tag{5.7}$$

where $\mathbf{e}_{\perp n}$ and $\mathbf{h}_{\perp n}$ are normalised transverse eigenfunctions of our waveguide defined exactly as in Marcuwitz and Schwinger, but where we define

$$\left.\begin{aligned} \mathbf{e}_{zn} &= [\nabla_\perp \cdot (\mathbf{h}_{\perp n} \times \hat{\mathbf{z}})]\hat{\mathbf{z}} = (\partial_x h_{yn} - \partial_y h_{xn})\hat{\mathbf{z}} \\ \mathbf{h}_{zn} &= [\nabla_\perp \cdot (\mathbf{e}_{\perp n} \times \hat{\mathbf{z}})]\hat{\mathbf{z}} = (\partial_x e_{yn} - \partial_y e_{xn})\hat{\mathbf{z}} \end{aligned}\right\} \tag{5.8}$$

to eliminate any presumption of exp ($j\omega t$) variation, $\hat{\mathbf{z}}$ being the unit vector in the z direction, we find on following much the same derivation as Marcuwitz and Schwinger:

$$\left.\begin{aligned} -\partial_z \partial_t V_n^{\mathrm{TE}} &= \mu_0 \partial_t^2 I_n^{\mathrm{TE}} \\ -\partial_z \partial_t I_n^{\mathrm{TE}} &= \left[\varepsilon_0 \partial_t^2 + \frac{1}{\mu_0}(k_{cn}^{\mathrm{TE}})^2\right] V_n^{\mathrm{TE}} + \partial_t \int_\Sigma \mathbf{e}_{\perp n}^{\mathrm{TE}} \cdot \mathbf{J}\, d\Sigma \\ -\partial_z \partial_t V_n^{\mathrm{TM}} &= \left[\mu_0 \partial_t^2 + \frac{1}{\varepsilon_0}(k_{cn}^{\mathrm{TM}})^2\right] I_n^{\mathrm{TM}} - \frac{1}{\varepsilon_0} \int_\Sigma J_z e_{zn}^{\mathrm{TM}}\, d\Sigma \\ -\partial_z \partial_t I_n^{\mathrm{TM}} &= \varepsilon_0 \partial_t^2 V_n^{\mathrm{TM}} + \partial_t \int_\Sigma \mathbf{e}_{\perp n}^{\mathrm{TM}} \cdot \mathbf{J}\, d\Sigma \\ V_n^{\mathrm{TE}} &= -\mu_0 \partial_t p_n^{\mathrm{TE}} \\ I_n^{\mathrm{TM}} &= \varepsilon_0 \partial_t q_n^{\mathrm{TM}} + \int_\Sigma J_z e_{zn}^{\mathrm{TM}}\, d\Sigma \end{aligned}\right\} \tag{5.9}$$

with TE, TM modes defined according to whether $\mathbf{e}_{zn} = e_{zn}\hat{\mathbf{z}}$, $\mathbf{h}_{zn} = h_{zn}\hat{\mathbf{z}}$ is *zero* or not. The k_{cn}s are the eigenvalues. The $d\Sigma$ integrals are taken over the cross-section of the waveguide.

All the problems involve the evaluation of these $d\Sigma$ *coupling integrals* and it is *only* in these that the *hybrid* method will differ from the simple *beam conformation* methods. We choose the Marcuwitz and Schwinger formulation precisely because of the simple form it gives to these integrals. The more popular Helmholtz formulation involves **curl** **J** and $\nabla\rho$ in these integrals: the present ones however are directly representative of the transverse and axial energy exchange. It is worth noting that in beam conformation methods there is some ambiguity in the definition of **J** because we do not have a density function, and the field is actually singular or discontinuous at the beam. In the hybrid method we are in a position to give a consistent maxwellian evaluation of **J** and hence the coupling integrals themselves.

We work in the simple phase space N. All the coupling integrals contain terms of the form

$$\int_{\Xi} e_{\xi i}(x, y)\, n\, v_{\xi}\, dv_x\, dv_y\, dv_z\, dx\, dy \qquad (\xi = x, y \text{ or } z)$$

for a density function n, to be consistent with the maxwellian definition of **J**, where the integration is over all velocities and over Σ (the waveguide cross-section) in x, y *at a fixed* z and t.

Presuming n to be known at the $z = z_1$ entry plane (hyperplane in N or P) for all t, we can define the *pull-back problem* as that of expressing n—and hence the integral—at an arbitrary z, t in terms of this entry plane distribution.

Central to the problem is the observation that the electrons in the 'current' z plane at the current t crossed the $z = z_1$ entry plane in general *at different* $t = t_1$ *times*. To treat this contingency correctly is very difficult, so we will assume that, at least to zero order, v_{z1} is *the same for all electrons*, or

$$n(\mathbf{x}, \mathbf{v}, t) \approx n(\mathbf{x}, v_{\perp}, \phi, t)\delta(v_z - v_{z1(0)})$$

where $v_{z1(0)}$ is a system constant.

Consider a slice of phase space (N) lying in a constant t hyperplane. Call it $\Xi(t)$. Each trajectory intersects $\Xi(t)$ in a single point. So each trajectory is also uniquely defined by the point $x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1}, t_1$ at which it intersects the $t = t_1$ hyperplane, and the total of these points defines $\Xi(t_1)$. So we can reparametrize the entire set of $\Xi(t)$s for arbitrary t in terms of $x_1, y_1, \ldots, v_{z1}$ at which each trajectory crosses $\Xi(t_1)$, and t itself: this is the familiar Hamilton–Jacobi canonical transformation to initial values. Write it thus:

$$x = x(x_1, y_1, \ldots, v_{z1}, t)$$

$$y = y(x_1, y_1, \ldots, v_{z1}, t)$$

$$\ldots\ldots\ldots$$

$$\ldots\ldots\ldots$$

$$z = z(x_1, y_1, \ldots, v_{z1}, t)$$

The importance of this transformation is that the total 'area' of the $\Xi(t)$ corresponding to $\Xi(t_1)$ at arbitrary t is constant (this is Liouville's theorem) if the system is

Hamiltonian, or

$$\int_{\Xi(t)} dx\ dy\ dz\ dv_x\ dv_y\ dv_z = \int_{\Xi(t_1)} dx\ dy\ dz\ dv_x\ dv_y\ dv_z$$

Relativistically, this does not hold for the observed velocities $\mathbf{v}$ but for the world velocity $\mathbf{u}$. Since

$$dv_x\ dv_y\ dv_z = \frac{1}{\gamma^5}\, du_x\ du_y\ du_z$$

the jacobian of the transform from x, y, z, v_x, v_y, v_z to $x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1}$ is not now 1 but γ_1^5/γ^5:

$$dx\ dy\ dz\ dv_x\ dv_y\ dv_z = (\gamma_1^5/\gamma^5)\ dx_1\ dy_1\ dz_1\ dv_{x1}\ dv_{y1}\ dv_{z1}$$

Now our coupling integrals contain instead the forms without dz, or $dx\ dy\ dv_z\ dv_y\ dv_z$, for fixed z, t. In terms of the transformation to the $t = t_1$ plane initial values, t being fixed so no dt integration terms appear, this takes the form

$$dx\ dy\ dv_x\ dv_y\ dv_z \equiv \alpha_{z1}\ dx_1\ dy_1\ dv_{x1}\ dv_{y1}\ dv_{z1} + \alpha_{y1}\ dx_1\ dz_1\ dv_{x1}\ dv_{y1}\ dv_{z1} + \ldots + \alpha_{v_{z1}}\ dx_1\ dy_1\ dz_1\ dv_{x1}\ dv_{y1}$$

for various jacobian coefficients $\alpha_{z1}, \alpha_{y1}, \ldots, \alpha_{v_{z1}}$ here labelled in terms of the 'missing' differential. But *under our constant v_z assumption*, the space over which we integrate being a constant z space at constant t, it will correspond to a space of constant z_1 at t_1. So all the integral terms involving dz_1 also drop out, and only the α_{z1} term survives. By appropriate use of Cramer's rule we can evaluate α_{z1} as

$$\alpha_{z1} = |J|\,\partial_z z_1$$

where $|J|$ is the jacobian of the entire 6×6 transform, which we know to be γ_1^5/γ^5 and $\partial_z z_1$ or $\partial z_1/\partial z$ is the variation of z_1 with z under the transformation. But again using our constant v_z assumption, $\partial_z z_1 \approx 1$, so $\alpha_{z1} \approx \gamma_1^5/\gamma^5$.

Now $x, y, z, v_x, v_y, v_z, t$ and $x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1}, t_1$ where $x = x(x_1, y_1, \ldots, v_{z1}, t)$, $y = y(x_1, y_1, \ldots, v_{z1}, t)$, etc., are *on the same trajectory*. But this does not mean we can equate the densities n at the two points because of the failure of Liouville's theorem in $\mathbf{x}, \mathbf{v}, t$ space. By the definition of n we must have:

$$\int_{\Xi(t)} n\ d\mathbf{x}\ d\mathbf{v} = \int_{\Xi(t_1)} n\ d\mathbf{x}_1\ d\mathbf{v}_1$$

but here it is $d\mathbf{x}\ d\mathbf{u} = \gamma^5\ d\mathbf{x}\ d\mathbf{v}$ that is the invariant differential form. So the scalar invariant is not n but n/γ^5. Hence $n(x, y, z, v_x, v_y, v_z, t) = (\gamma^5/\gamma_1^5) \cdot n(x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1}, t_1)$ and clearly this (γ^5/γ_1^5) cancels with the jacobian (γ_1^5/γ^5) in our coupling integrals to give the form:

$$\int e_{\xi i}[x(x_1, \ldots, t), y(x_1, \ldots, t)]v_\xi(x_1, \ldots, t)n(x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1}, t_1) \times dv_{x1}\ dv_{y1}\ dv_{z1}\ dx_1\ dy_1$$

where all terms are evaluated for $(x_1, \ldots, t)$ or $(x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1}, t)$ except n, which is for $x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1}$ *and* t_1.

Under our constant v_z assumption, all electrons in the same z plane at the same t will pass through the same z_1 plane at the same t_1; so, choosing t_1 to take us back

precisely to the entry plane $z = z_1$ for whatever z, t at which we are performing the integration by the requirement that

$$t_1 = t - (z - z_1)/v_{z1}$$

we recover our original entry-plane related transformation from the Hamilton–Jacobi one. Further, though the choice of the t_1 space being dependent on z and t, the actual transformation itself is implicity dependent on z and t, and so this dependence is given to the coupling integrals overall.

We can make further simplifications: the entry-plane is a system constant, so z_1 can disappear explicitly. The dv_{z1} integration is trivial because of the δ function form of the v_{z1} distribution, so that can go. Let us further assume that the entry-plane distribution is independent of t (so t_1 now) so that n reduces to $n(x_1, y_1, v_{x1}, v_{y1})$ alone. Finally, let us assume that $x(x_1, \ldots, t)$, $y(x_1, \ldots, t)$ vary only over a small Larmor circle, so that $e_{\xi i}[x(x_1, \ldots, t), y(x_1, \ldots, t)]$ can be approximated by $e_{\xi i}(x_1, y_1)$ at whatever z, t. This is consistent with our zero-order solution if the Larmor radius is small compared with the tube diameter; it is tantamount to considering operation at the fundamental cyclotron harmonic only.

Under these assumptions, *all z, t dependence of the coupling integrals will now reside in v_ξ*. This contrasts with the Vlasov methods in which the dynamical dependence is coupled into the EM equations though the density function in the same coupling integrals. The present analysis will proceed by expressing v_ξ linearly in the field quantities $V_n(z, t)$, $I_n(z, t)$, $q_n(z, t)$, $p_n(z, t)$ by means of the equations of motion, expanding E_x, E_y, E_z, B_x, B_y, B_z therein: these do not depend on x_1 y_1, v_{x1}, v_{y1} and so separate out from the integration of the coupling integrals, which fact will enable us to recover a dispersion relation.

We will also replace v_{x1}, v_{y1} by $v_{\perp 1}$, ϕ_1 under the definition $v_{x1} = v_{\perp 1} \sin \phi_1$, $v_{y1} = v_{\perp 1} \cos \phi_1$; this ϕ_1 is also related to the *zero-order* $\phi^{(0)}$ (rather than the 'actual' ϕ) at the 'current' z by

$$\phi^{(0)} = \phi_1 + \frac{\omega_{ce1}}{v_{z1}}(z - z_1)$$

6. Fourier analysis

Rather than postulate $\exp(j\omega t - jkz)$ dependence of our solutions, we take a full Fourier analysis. This enables us rigorously to dispose of the periodicity of the zero-order components.

We return to the space P and to our dynamical equations with $\mathbf{x}^{(\bar{0})}$ approximated by $\mathbf{x}^{(1)}$. We parametrize $\mathscr{T}$ (actually $\mathscr{T}^{(1)}$) by its zero-order components, choosing $x^{(0)}$, $y^{(0)}$, $z^{(0)}$, $v_\perp^{(0)}$, $v_z^{(0)} = v_{z1}$, $\phi^{(0)}$ and t. This is different to the approach used in Vlasov methods in that they use the 'actual' variables rather than the zero-order components. Other than this, our approach will closely parallel that of Baldwin *et al.* (1969). The objective is to reformulate the *products* in the dynamical equations as *convolution integrals* which will then give tractable terms on Fourier transformation.

$\phi^{(0)}$ here is defined by $v_x^{(0)} = v_\perp^{(0)} \sin \phi^{(0)}$, $v_y^{(0)} = v_\perp^{(0)} \cos \phi^{(0)}$. Consider a particular ('current') point in $\mathscr{T}$ (identified by the subscript 0) labelled by the coordinates $x_0^{(0)}$, $y_0^{(0)}$, $z_0^{(0)}$, $v_{\perp 0}^{(0)}$, $\phi_0^{(0)}$, t_0. The zero-order parts of the unique trajectory passing through this point (or its projection onto $\mathscr{T}^{(0)}$) can be described by the equations in the

parameter z':

$$\left.\begin{aligned} z^{(0)} &= z' \\ t &= t_0 - (z_0^{(0)} - z')/v_{z1} \\ \phi^{(0)} &= \phi_0^{(0)} - (z_0^{(0)} - z')\omega_{ce1}/v_{z1} \\ v_\perp^{(0)} &= v_{\perp 0}^{(0)} \end{aligned}\right\} \tag{6.1}$$

and so

$$\left.\begin{aligned} v_x^{(0)} &= v_{\perp 0}^{(0)} \sin\,[\phi_0^{(0)} - (z_0^{(0)} - z')\omega_{ce1}/v_{z1}] \\ v_y^{(0)} &= v_{\perp 0}^{(0)} \cos\,[\phi_0^{(0)} - (z_0^{(0)} - z')\omega_{ce1}/v_{z1}] \end{aligned}\right\} \tag{6.2}$$

all independent of $x_0^{(0)}$, $y_0^{(0)}$ and, but for t, of t_0 too. Next we integrate along the trajectories with respect to z' from the datum entry plane $z' = z_1$ to the present $z' = z_0^{(0)}$.

Denote the t value corresponding to z' as t': $t' = t_0 - (z_0^{(0)} - z')/v_{z1}$. Now the first-order variables may depend on z' and on $z_0^{(0)}$, t_0 through t' as well as $\phi_0^{(0)}$ and $v_0^{(0)}$ (and of course $x_0^{(0)}$, $y_0^{(0)}$).

Hence, on integration, products of zero-order and first-order variables will appear as

$$\int_{z_1}^{z_0^{(0)}} \phi^{(1)}(z', t')\psi^{(0)}(z_0^{(0)} - z')\,dz' = \int_{z_1}^{z_0^{(0)}} \phi^{(1)}\left(z', t_0 - \frac{z_0^{(0)} - z'}{v_{z1}}\right)\psi^{(0)}(z_0^{(0)} - z')\,dz'$$

The $d\chi^{(1)}/d\tau$ terms, on the other hand, now appear as exact integrals because of the choice of path of integration along the trajectory (τ) and, by $dz^{(0)}/d\tau = v_{z1}$ if τ is uniform with t, and since $z^{(0)} = z'$ along the trajectory

$$\int_{z_1}^{z_0^{(0)}} \frac{d}{d\tau}\chi^{(1)}\,dz' = v_{z1}\int_{z_1}^{z_0^{(0)}} \frac{d}{d\tau}\chi^{(1)}\,d\tau = v_{z1}\int_{z_1}^{z_0^{(0)}} d(\chi^{(1)})$$

but in the datum entry plane, $\chi^{(1)}(z_1) = 0$ by definition, so these reduce to just $v_{z1}\chi^{(1)}(z_0^{(0)})$.

Hence the first equation of motion takes the form:

$$\begin{aligned} v_{z1}v_x^{(1)}(z_0^{(0)}, t_0) + \omega_{ce1}v_{z1}y^{(1)}(z_0^{(0)}, t_0) & \\ - \frac{\omega_{ce1}}{\gamma_1}\int v_y^{(0)}(z_0^{(0)} - z')\gamma^{(1)}(z', t')\,dz' & \\ = \frac{e}{m_0\gamma_1 c^2}\sum_i \int E_i v_i^{(0)}(z_0^{(0)} - z')v_x^{(0)}(z_0^{(0)} - z')\,dz' & \\ - \frac{e}{m_0\gamma_1}\int [E_x + v_y^{(0)}(z_0^{(0)} - z')B_z - v_{z1}B_y]\,dz' & \end{aligned} \tag{6.3}$$

on substituting $v_y^{(1)} = dy^{(1)}/d\tau$ in the second term.

The field terms can now be brought into a convolution form too by remembering that

$$E_\xi = \sum_n V_n e_{\xi n} \quad \text{for} \quad \xi = x \text{ or } y, \quad E_z = \sum_n q_n e_{zn}$$

and putting

$$e_{\zeta n} \approx e_{\xi n}(x^{(0)}, y^{(0)}) = e_{\xi n}[x^{(0)}(z_0^{(0)} - z'), y^{(0)}(z_0^{(0)} - z')]$$

and

$$V_n = V_n(z', t'), \qquad q_n = q_n(z', t')$$

we see that the RHS field terms above are again convolutions.

We now take Fourier–Laplace transforms with respect to the 'current point' variables $z_0^{(0)}$ and t_0. In t (i.e. t_0) we take the usual $\int_0^\infty$ transform to ω. But for z it is patently inconsistent with our physical model to take $\int_{-\infty}^{\infty}$ integration. So instead we again use a one-sided transform, integrating $\int_{z_1}^{\infty}$ to k. We will take the transform for *real* k and for Im $(\omega) > 0$ so our results will remain consistent with the analysis of Briggs (1964) and Bers (1975). We can then show that, for $\phi^{(1)}(z', t') = 0$ for $t' < 0$:

$$\begin{aligned}\int_0^\infty \int_{z_1}^\infty \int_{z_1}^{z_0^{(0)}} \phi^{(1)}\left(z', t_0 - \frac{z_0^{(0)} - z'}{v_{z1}}\right)\psi^{(0)}(z_0^{(0)} - z')\, dz' \\ \times \exp(-jkz_0^{(0)}) \exp(j\omega t_0)\, dz_0^{(0)}\, dt_0 \\ = \int_0^\infty \int_{z_1}^\infty \phi^{(1)}(z', t') \exp(-jkz') \exp(j\omega t')\, dz'\, dt' \\ \times \int_0^\infty \psi^{(0)}(u) \exp\left[-j\left(k - \frac{\omega}{v_{z1}}\right)u\right] du \end{aligned} \tag{6.4}$$

a slightly modified form of the usual Fourier convolution theorem. Since we integrate for real k and Im $(\omega) > 0$, $\exp[-j(k - (\omega/v_{z1}))u]$ goes to zero as u goes to $+\infty$ in the usual Laplace manner, so the $\psi^{(0)}$ integrals are easily evaluated knowing the $v_x^{(0)}$, $v_y^{(0)}$, etc., functions. For the field terms we obtain products of the form

$$\int V_n(z', t') \exp(-jkz') \exp(j\omega t')\, dz'\, dt' \times \int e_{\xi n}(x^{(0)}(u), y^{(0)}(u)) \exp\left[-j\left(k - \frac{\omega}{v_{z1}}\right)u\right] du$$

and it is consistent with our earlier assumptions about Larmor radii to reckon $e_{\xi n}(x^{(0)}(u), y^{(0)}(u)) \approx e_{\xi n}(x_0^{(0)}, y_0^{(0)})$, independent of u, so we obtain

$$V_n(k, \omega) e_{\xi n}(x_0^{(0)}, y_0^{(0)}) \frac{1}{j\left(k - \frac{\omega}{v_{z1}}\right)}$$

Using $\cos x = \frac{1}{2}[\exp(jx) + \exp(-jx)]$ and $\sin x = (1/2j)[\exp(jx) - \exp(-jx)]$ to express $v_x^{(0)}$, $v_y^{(0)}$ along the trajectory as:

$$v_x^{(0)} = \frac{v_{\perp 0}^{(0)}}{2j}[\exp(j\phi_0^{(0)}) \exp(-jk_c u) - \exp(-j\phi_0^{(0)}) \exp(jk_c u)]$$

$$v_y^{(0)} = \frac{v_{\perp 0}^{(0)}}{2}[\exp(j\phi_0^{(0)}) \exp(-jk_c u) + \exp(-j\phi_0^{(0)}) \exp(jk_c u)]$$

where $u = z_0^{(0)} - z'$, corresponding to the new variable of integration, and we define $k_c = \omega_{ce1}/v_{z1}$ for abbreviation (we also use $k_e = \omega/v_{z1}$ later), and defining:

$$\beta_0 = \frac{1}{j(k - k_e)}, \qquad \beta_1^{\pm} = \frac{1}{j[k_c \pm (k - k_e)]}, \qquad \beta_2^{\pm} = \frac{1}{j[2k_c \pm (k - k_e)]}$$

we obtain for the Fourier–Laplace transform of our first equation:

$$\begin{aligned} v_x^{(1)}(k, \omega) + \omega_{ce1} y^{(1)}(k, \omega) - \frac{\omega_{ce1}}{v_{z1}\gamma_1} \frac{v_\perp}{2} [\beta_1^+ \exp(j\phi) - \beta_1^- \exp(-j\phi)]\gamma^{(1)}(k, \omega) \\ = \frac{e}{m_0 \gamma_1 c^2 v_{z1}} \sum_n \left\{ V_n(k, \omega) \frac{v_\perp^2}{4} \left(e_{xn}[2\beta_0 - \beta_2^+ \exp(j2\phi) + \beta_2^- \exp(-j2\phi)] \right.\right. \\ \left. + \frac{e_{yn}}{j}[\beta_2^+ \exp(j2\phi) + \beta_2^- \exp(-j2\phi)] \right) \\ \left. + q_n(k, \omega) v_{z1} e_{zn} \frac{v_\perp}{2j} [\beta_1^+ \exp(j\phi) + \beta_1^- \exp(-j\phi)] \right\} \\ - \frac{e}{m_0 \gamma_1 v_{z1}} \sum_n \left\{ V_n(k, \omega) e_{xn} \beta_0 + p_n(k, \omega) h_{zn} \frac{v_\perp}{2} \right. \\ \left. \times [\beta_1^+ \exp(j\phi) - \beta_1^- \exp(-j\phi)] - v_{z1} I_n(k, \omega) h_{yn} \beta_0 \right\} \end{aligned} \tag{6.5}$$

where, and from now on, we scrap the cumbersome $^{(0)}_{0}$ notation (although the zero-order definition of these terms should be borne in mind) writing just $v_\perp$ for $v_{\perp 0}^{(0)}$, z for $z_0^{(0)}$, ϕ for $\phi_0^{(0)}$ etc.

Now the equations

$$\frac{d}{d\tau} x^{(1)} = v_x^{(1)}, \qquad \frac{d}{d\tau} y^{(1)} = v_y^{(1)}$$

do *not* admit of a formulation in terms of exact differentials and convolutions alone. Instead, our Fourier–Laplace transform gives us, from $d/d\tau = \omega_{ce1}\partial_\phi + v_{z1}\partial_z + \partial_t$ (exactly in these zero-order variables):

$$\left.\begin{aligned} [-j(\omega - kv_{z1}) + \omega_{ce1}\partial_\phi]x^{(1)}(k, \omega) = v_x^{(1)}(k, \omega) \\ [-j(\omega - kv_{z1}) + \omega_{ce1}\partial_\phi]y^{(1)}(k, \omega) = v_y^{(1)}(k, \omega) \end{aligned}\right\} \tag{6.6}$$

To eliminate the ∂_ϕs, we take a Fourier transform in ϕ to a (discrete) variable s by

$$\int_0^{2\pi} (\,\cdot\,) \exp(-js\phi)\, d\phi$$

which is of course a Fourier series in ϕ. This finally gives, writing

$$f(\phi) = \sum_s f_{(s)} \cdot \exp(js\phi)/2\pi$$

the full set of transformed equations in k, ω, s (the δs here are Kronecker deltas):

$$v^{(1)}_{x(s)}(k,\omega)+\omega_{ce1}y^{(1)}_{(s)}(k,\omega)-\frac{\omega_{ce1}}{\gamma_1 v_{z1}}\frac{v_\perp}{2}(\beta_1^+\gamma^{(1)}_{(s-1)}(k,\omega)-\beta_1^-\gamma^{(1)}_{(s+1)}(k,\omega))$$
$$=\frac{2\pi e}{m_0\gamma_1 c^2 v_{z1}}\sum_n\left[V_n(k,\omega)\frac{v_\perp^2}{4}\left(e_{xn}(2\beta_0\delta_{s0}-\beta_2^+\delta_{s2}+\beta_2^-\delta_{s(-2)})\right.\right.$$
$$\left.+\frac{e_{yn}}{j}(\beta_2^+\delta_{s2}+\beta_2^-\delta_{s(-2)})\right)$$
$$+q_n(k,\omega)v_{z1}e_{xn}\frac{v_\perp}{2j}(\beta_1^+\delta_{s1}+\beta_1^-\delta_{s(-1)})\Bigg]$$
$$-\frac{2\pi e}{m_0\gamma_1 v_{z1}}\sum_n\left[V_n(k,\omega)e_{xn}\beta_0\delta_{s0}+p_n(k,\omega)h_{zn}\frac{v_\perp}{2}\right.$$
$$\left.\times(\beta_1^+\delta_{s1}-\beta_1^-\delta_{s(-1)})-v_{z1}I_n(k,\omega)h_{yn}\beta_0\delta_{s0}\right] \tag{6.7}$$

$$v^{(1)}_{y(s)}(k,\omega)-\omega_{ce1}x^{(1)}_{(s)}(k,\omega)+\frac{\omega_{ce1}}{\gamma_1 v_{z1}}\frac{v_\perp}{2j}(\beta_1^+\gamma^{(1)}_{(s-1)}(k,\omega)-\beta_1^-\gamma^{(1)}_{(s+1)}(k,\omega))$$
$$=\frac{2\pi e}{m_0\gamma_1 c^2 v_{z1}}\sum_n\left[V_n(k,\omega)\frac{v_\perp^2}{4}\left(\frac{e_{xn}}{j}(\beta_2^+\delta_{s2}+\beta_2^-\delta_{s(-2)})\right.\right.$$
$$\left.+e_{yn}(2\beta_0\delta_{s0}+\beta_2^+\delta_{s2}-\beta_2^-\delta_{s(-2)})\right)$$
$$+q_n(k,\omega)v_{z1}e_{zn}\frac{v_\perp}{2}(\beta_1^+\delta_{s1}-\beta_1^-\delta_{s(-1)})\Bigg]$$
$$-\frac{2\pi e}{m_0\gamma_1 v_{z1}}\sum_n\left[V_n(k,\omega)e_{yn}\beta_0\delta_{s0}+v_{z1}I_n(k,\omega)h_{xn}\beta_0\delta_{s0}\right.$$
$$\left.-p_n(k,\omega)h_{zn}\frac{v_\perp}{2j}(\beta_1^+\delta_{s1}+\beta_1^-\delta_{s(-1)})\right] \tag{6.8}$$

$$v^{(1)}_{z(s)}(k,\omega)=\frac{2\pi e}{m_0\gamma_1 c^2}\sum_n\left[V_n(k,\omega)\frac{v_\perp}{2}\left(\frac{e_{xn}}{j}(\beta_1^+\delta_{s1}+\beta_1^-\delta_{s(-1)})\right.\right.$$
$$\left.\left.+e_{yn}(\beta_1^+\delta_{s1}-\beta_1^-\delta_{s(-1)})\right)+q_n(k,\omega)v_{z1}e_{zn}\beta_0\delta_{s0}\right]$$
$$-\frac{2\pi e}{m_0\gamma_1 v_{z1}}\sum_n\left[q_n(k,\omega)e_{zn}\beta_0\delta_{s0}+\frac{v_\perp}{2}I_n(k,\omega)\right.$$
$$\left.\times\left(\frac{h_{yn}}{j}(\beta_1^+\delta_{s1}+\beta_1^-\delta_{s(-1)})-h_{xn}(\beta_1^+\delta_{s1}-\beta_1^-\delta_{s(-1)})\right)\right] \tag{6.9}$$

$$\gamma^{(1)}_{(s)}(k,\omega)=-\frac{2\pi e}{m_0 v_{z1}c^2}\sum_n\left[V_n(k,\omega)\frac{v_\perp}{2}\left(\frac{e_{xn}}{j}(\beta_1^+\delta_{s1}+\beta_1^-\delta_{s(-1)})\right.\right.$$
$$\left.\left.+e_{yn}(\beta_1^+\delta_{s1}-\beta_1^-\delta_{s(-1)})\right)+q_n(k,\omega)v_{z1}e_{zn}\beta_0\delta_{s0}\right] \tag{6.10}$$

$$-j(\omega-kv_{z1}-s\omega_{ce1})x^{(1)}_{(s)}(k,\omega)=v^{(1)}_{x(s)}(k,\omega) \tag{6.11}$$

$$-j(\omega-kv_{z1}-s\omega_{ce1})y^{(1)}_{(s)}(k,\omega)=v^{(1)}_{y(s)}(k,\omega) \tag{6.12}$$

The $\phi \rightarrow s$ transformation is very revealing, for it is precisely the appearance of 'odd' coefficients here that represents the corrugation or bunching that we are looking for: a regular sinusoidal function (as $v_x^{(0)}$, $v_y^{(0)}$) will show only balanced $s = +1$ and $s = -1$ components in this cartesian representation.

To obtain the dispersion equation, we put these as $v_\xi^{(0)} + v_\xi^{(1)}$ into the coupling integrals, considered at length in the preceding section. To our earlier simplifying assumptions we now add that the entry-plane distribution n_1 (i.e. n at $z = z_1$) is independent of ϕ_1 and hence of the *zero-order* $\phi_0^{(0)}$ or 'ϕ'. This enables us to express the coupling integrals in the form:

$$\int e_{\xi i}(x, y) n_1(x, y, v_\perp) v_\xi(x, y, v_\perp, z, t, \phi)\, d\phi\, v_\perp\, dx\, dy\, dv_\perp$$

remembering $dv_x\, dv_y = v_\perp\, d\phi\, dv_\perp$. Observe that now all the z, t, ϕ dependence is in v_ξ.

Furthermore, putting $v_\xi = v_\xi^{(0)} + v_\xi^{(1)}$ we have $v_z^{(0)} = v_{z1}$ and $v_x^{(0)} = v_{\perp 0}^{(0)} \sin \phi_0^{(0)} = v_\perp \sin \phi$, $v_y^{(0)} = v_\perp \cos \phi$ so these latter two integrate $d\phi$ to zero, and the v_{z1} term just gives the constant:

$$\int e_{\xi i}(x, y) n_1(x, y, v_\perp) v_{z1}\, 2\pi v_\perp\, dx\, dy\, dv_\perp$$

This constant represents the static field due to the presence of the static DC beam: it contributes only constant V_n, I_n particular integrals and evidently, being an independent term, does not contribute to the dispersion equation. So we are left with just the $v_\xi^{(1)}$ terms. Now

$$v_\xi^{(1)} = \frac{1}{2\pi} \sum_s v_{\xi(s)}^{(1)} \exp(js\phi)$$

and, on integrating $d\phi$, *all these terms go to zero except the* $s = 0$ *term.* This elimination of higher cyclotron harmonics is a consequence of our earlier assumption about the constancy of $e_{\xi n}$ across a Larmor diameter: we are here developing a theory for interaction just at the fundamental.

Since $\int_0^{2\pi} 1\, d\phi = 2\pi$, we are left with:

$$\mathscr{F}\mathscr{L}\left(\int e_{\xi i} J_\xi\, d\Sigma\right) = \int e_{\xi i}(x, y) n_1(x, y, v_\perp) v_{\xi(s=0)}^{(1)}(x, y, v_\perp, k, \omega)\, v_\perp\, dx\, dy\, dv_\perp$$

as the Fourier–Laplace transform of the coupling integral components.

For $s = 0$,

$$v_{z(s=0)}^{(1)}(k, \omega) = -2\pi j \frac{e}{m_0 \gamma_1} \left(1 - \frac{v_{z1}^2}{c^2}\right) \frac{1}{(\omega - k v_{z1})} \sum_n q_n e_{zn}$$

since $\beta_0 = 1/j(k - k_e) = jv_{z1}/(\omega - kv_{z1})$, so that

$$\mathscr{F}\mathscr{L}\left(\int e_{zm} J_z\, d\Sigma\right) = -2\pi j \frac{e}{m_0} \left(1 - \frac{v_{z1}^2}{c^2}\right) \frac{1}{(\omega - k v_{z1})} \sum_n q_n(k, \omega) \mathrm{EE}_{zn}^{zm} \qquad (6.13)$$

where we define

$$\mathrm{EE}_{zn}^{zm} = \int e_{zm} e_{zn} \frac{1}{\gamma_1} n_1(x, y, v_\perp)\, v_\perp \, dx\, dy\, dv_\perp \tag{6.14}$$

keeping γ_1 under the integral because it depends on $v_\perp = v_{\perp 1}$.

The transverse terms are a little more subtle. We need to use

$$\gamma_{(\pm 1)}^{(1)} = -\frac{2\pi e}{m_0 c^2 v_{z1}} \sum \left[V_n(k, \omega) \frac{v_\perp}{2} \left(\frac{e_{xn}}{j} \pm e_{yn} \right) \right] \beta_1^{\pm}$$

and to substitute for $x_{(s=0)}^{(1)}$, $y_{(s=0)}^{(1)}$ in the $v_{x(s=0)}^{(1)}$, $v_{y(s=0)}^{(1)}$ equations; but then we encounter a small problem: to abstract the dispersion relation, we want all terms in k, ω, including $[(\omega - kv_{z1}) \pm \omega_{ce1}]$ to come outside the $dx\, dy\, dv_\perp$ integration. But ω_{ce1} (the zero-order ω_c or $eB_0/m_0\gamma_1$) is dependent on $v_\perp$ or $v_{\perp 1}$. We can justify bringing this outside the integration if we suppose that the spread of $v_\perp(v_{\perp 1})$ is not large. So, with the v_{z1} assumption earlier (although the present requirement is arguably less stringent), this means that our theory is restricted to *narrow velocity distributions.*

With that assumption, but still including the γ_1s in our integrals, we define:

$$\left.\begin{aligned} \mathrm{EE}_{yn}^{xm(\mathrm{rel})} &= \int e_{xm} e_{yn} \frac{v_\perp^2}{\gamma_1 c^2} n_1\, v_\perp \, dx\, dy\, dv_\perp \\ \mathrm{EE}_{yn}^{xm(\mathrm{f})} &= \int e_{xm} e_{yn} \frac{n_1}{\gamma_1} v_\perp \, dx\, dy\, dv_\perp \\ \mathrm{EH}_{yn}^{xm} &= \int e_{xm} h_{yn} \frac{n_1}{\gamma_1} v_\perp \, dx\, dy\, dv_\perp \end{aligned}\right\} \tag{6.15}$$

and likewise $\mathrm{EE}_{xn}^{xm(\mathrm{rel})}$, $\mathrm{EE}_{xn}^{xm(\mathrm{f})}$, EH_{xn}^{xm} etc., to obtain:

$$\begin{aligned} \mathscr{F}\mathscr{L}\left(\int e_{xm} J_x \, d\Sigma \right) = \frac{(\omega - kv_{z1})^2}{(\omega - kv_{z1})^2 - \omega_{ce1}^2} \Bigg[& \omega_{ce1} \frac{\pi e}{m_0} \frac{1}{((\omega - kv_{z1})^2 - \omega_{ce1}^2)} \\ & \times \sum_n V_n \mathrm{EE}_{yn}^{xm(\mathrm{rel})} - 2\pi j \frac{e}{m_0} \frac{1}{(\omega - kv_{z1})} \\ & \times \sum_n (V_n(\mathrm{EE}_{xn}^{xm(\mathrm{f})} - \tfrac{1}{2}\mathrm{EE}_{xn}^{xm(\mathrm{rel})}) - v_{z1} I_n \mathrm{EH}_{yn}^{xm}) \Bigg] \\ - \frac{\omega_{ce1}(\omega - kv_{z1})}{j((\omega - kv_{z1})^2 - \omega_{ce1}^2)} \Bigg[& \omega_{ce1} \frac{\pi e}{m_0} \frac{1}{((\omega - kv_{z1})^2 - \omega_{ce1}^2)} \\ & \times \sum_n V_n \mathrm{EE}_{xn}^{xm(\mathrm{rel})} + 2\pi j \frac{e}{m_0} \frac{1}{(\omega - kv_{z1})} \\ & \times \sum_n (V_n(\mathrm{EE}_{yn}^{xm(\mathrm{f})} - \tfrac{1}{2}\mathrm{EE}_{yn}^{xm(\mathrm{rel})}) + v_{z1} I_n \mathrm{EH}_{xn}^{xm}) \Bigg] \end{aligned} \tag{6.16}$$

$$\mathscr{F}\mathscr{L}\left(\int e_{ym} J_y \, d\Sigma\right) = \frac{(\omega - kv_{z1})^2}{(\omega - kv_{z1})^2 - \omega_{ce1}^2}\left[-\omega_{ce1}\frac{\pi e}{m_0}\frac{1}{((\omega - kv_{z1})^2 - \omega_{ce1}^2)}\right.$$
$$\times \sum_n V_n \mathrm{EE}_{xn}^{ym(\mathrm{rel})} - 2\pi j \frac{e}{m_0}\frac{1}{(\omega - kv_{z1})}$$
$$\left.\times \sum_n (V_n(\mathrm{EE}_{yn}^{ym(\mathrm{f})} - \tfrac{1}{2}\mathrm{EE}_{yn}^{ym(\mathrm{rel})}) + v_{z1} I_n \mathrm{EH}_{xn}^{ym})\right]$$
$$- \frac{\omega_{ce1}(\omega - kv_{z1})}{j((\omega - kv_{z1})^2 - \omega_{ce1}^2)}\left[\omega_{ce1}\frac{\pi e}{m_0}\frac{1}{((\omega - kv_{z1})^2 - \omega_{ce1}^2)}\right.$$
$$\times \sum_n V_n \mathrm{EE}_{yn}^{ym(\mathrm{rel})} - 2\pi j \frac{e}{m_0}\frac{1}{(\omega - kv_{z1})}$$
$$\left.\times \sum_n (V_n(\mathrm{EE}_{xn}^{ym(\mathrm{f})} - \tfrac{1}{2}\mathrm{EE}_{xn}^{ym(\mathrm{rel})}) - v_{z1} I_n \mathrm{EH}_{yn}^{ym})\right] \tag{6.17}$$

where the (rel) integral terms derive ultimately from the relativistic terms in the equations of motion, the EH integrals from the $\mathbf{v} \times \mathbf{B}$ terms. These expressions remain very general: neither a particular waveguide cross-section nor a particular zero-order electron distribution has been assumed, although we have imposed velocity-space restrictions.

So finally we need but to take the Fourier–Laplace transforms of the remaining parts of the V_n, I_n EM equations. The one way in which our transforms are 'non-standard' is that we use the one-sided $\int_{z_1}^{\infty}$ Fourier transform for z. The EM equations contain a $\partial_z(\partial/\partial z)$ operator and so this will now generate terms at the z_1 boundary. This is a convenient way to introduce the *driving* terms representing the *input waveguide* of the TWT gyrotron, which we now take to be at (or just left of) $z = z_1$. If we assume that the presence of the input waveguide makes V_n and I_n approximately sinusoidal in time at the $z = z_1$ boundary, and so likewise $\partial_t V_n$, $\partial_t I_n$ then these can be represented by exponentials like:

$$g^+ \exp(j\omega_e t) + g^- \exp(-j\omega_e t)$$

where ω_e is the *excitation frequency*. Now for arbitrary f, we have:

$$\int_{z_1}^{\infty} \partial_z f \exp(-jkz)\, dz = -f(z_1)\exp(-jkz_1) + jk\int_{z_1}^{\infty} f \exp(-jkz)\, dz$$

so that if we take $g_{V_n}^+ \exp(j\omega_e t) + g_{V_n}^- \exp(-j\omega_e t)$ as the z_1 boundary value of $\partial_t V_n$, the Fourier–Laplace transform of our entire set gives:

$$-k\omega V_n^{\mathrm{TE}} + \mu_0 \omega^2 I_n^{\mathrm{TE}} = \exp(-jkz_1)[g_{V_n}^{\mathrm{TE}+} j/(\omega + \omega_e) + g_{V_n}^{\mathrm{TE}-} j/(\omega - \omega_e)] \tag{6.18}$$

$$-k\omega I_n^{\mathrm{TE}} + \left[\varepsilon_0 \omega^2 - \frac{(k_{cn}^{\mathrm{TE}})^2}{\mu_0}\right] V_n^{\mathrm{TE}} + j\omega \mathscr{F}\mathscr{L}\left(\int \mathbf{e}_{\perp n}^{\mathrm{TE}} \cdot \mathbf{J}\, d\Sigma\right)$$
$$= \exp(-jkz_1)[g_{I_n}^{\mathrm{TE}+} j/(\omega + \omega_e) + g_{I_n}^{\mathrm{TE}-} j/(\omega - \omega_e)] \tag{6.19}$$

$$-k\omega V_n^{\mathrm{TM}} + \left[\mu_0 \omega^2 - \frac{(k_{cn}^{\mathrm{TM}})^2}{\varepsilon_0}\right] I_n^{\mathrm{TM}} + \frac{1}{\varepsilon_0}\mathscr{F}\mathscr{L}\left(\int e_{zn}^{\mathrm{TM}} J_z\, d\Sigma\right)$$
$$= \exp(-jkz_1)[g_{V_n}^{\mathrm{TM}+} j/(\omega + \omega_e) + g_{V_n}^{\mathrm{TM}-} j/(\omega - \omega_e)] \tag{6.20}$$

$$-k\omega I_n^{\mathrm{TM}} + \varepsilon_0 \omega^2 V_n^{\mathrm{TM}} + j\omega \mathscr{F}\mathscr{L}\left(\int \mathbf{e}_{\perp n}^{\mathrm{TM}} \cdot \mathbf{J}\, d\Sigma\right)$$
$$= \exp(-jkz_1)[g_{I_n}^{\mathrm{TM}+} j/(\omega + \omega_e) + g_{I_n}^{\mathrm{TM}-} j/(\omega - \omega_e)] \tag{6.21}$$

$$V_n^{\mathrm{TE}} = j\omega\mu_0 p_n^{\mathrm{TE}}$$

$$I_n^{\mathrm{TM}} = -j\omega\varepsilon_0 q_n^{\mathrm{TM}} + \mathscr{F}\mathscr{L}\left(\int e_{zn}^{\mathrm{TM}} J_z \, d\Sigma\right)$$

where now $V_n = V_n(k, \omega)$, $I_n = I_n(k, \omega)$, $q_n = q_n(k, \omega)$, $p_n = p_n(k, \omega)$, the $\mathscr{F}\mathscr{L}(\int \mathbf{e} \cdot \mathbf{J} \, d\Sigma)$ terms are as given previously and hence linear is all the V_i, I_i, q_i, p_i. Observe that the exp $(-jkz_1)$ factors will restore the $H(z - z_1)$ Heaviside step on inversion of the transforms, and the g^+ and g^- factors will ensure that the result is real. These $g^{\pm}$ factors now form a *discrete* distribution over the mode numbers, representing an EM entry-plane *datum* distribution analogous to the entry-plane electron density distribution which is of course continuous.

The *driver* terms now appear in a form directly analogous to the $\delta(z - z_1)$ exp $(j\omega_e t)$ excitation model used by Briggs (1964) or Bers (1975). In Bers' notation, any resulting dispersion equation will in our case give the right-hand $k_u(\omega)$ functions because we use the one-sided transform. It is amplification of such a right-hand wave that we are now looking for; a left-hand wave just travels into the gun and what happens there is rather unpredictable!

7. The dispersion equation

This full set of equations remains very general. However, we still have an infinite number of variables. The final simplification is that usually made, we think mistakenly, *first*. This is the 'single-mode assumption'. Now there is no reason to postulate *a priori* that the modes are 'unlinked'. Indeed, in the presence of the beam and hence the $\int \mathbf{e} \cdot \mathbf{J} \, d\Sigma$ coupling integrals, the nature of the dispersion equation is quite altered: every mode is, at least weakly, coupled to every other, and so the very concept of TE and TM mode identity should not be presumed. All we have now defining each mode is the basic $\mathbf{e}_{\perp n}$, $\mathbf{h}_{\perp n}$ eigenfunction. Exactly how the V_ns and I_ns for each mode behave with z and t and the dispersion relation involved is quite unknown: that is precisely what we are trying to find.

Regarding the equations as a matrix acting on the $\{V_n\}$, $\{I_n\}$ vector of variables, what we want is that 'off-diagonal' elements are generally negligible. Now these off-diagonal (relating mode m and mode n) elements are given entirely by the $\mathrm{EE}_{\psi n}^{\xi m}$, $\mathrm{EH}_{\psi n}^{\xi m}$ integrals. Note that because $\mathbf{h}_{\perp n} = \hat{\mathbf{z}} \times \mathbf{e}_{\perp n}$ so $h_{yn} = e_{xn}$, $h_{xn} = -e_{yn}$, we can always formulate the EH ones as EE ones. So quite simply, we want all $\mathrm{EE}_{\psi n}^{\xi m}$ to be small if $m \neq n$.

We consider this view to be integral to our approach: rather than *presume* a configuration and then see by calculation whether $\mathrm{EE}_{\psi n}^{\xi m} \approx \delta_{mn} \mathrm{EE}_{\psi n}^{\xi m}$ the basic objective of gyrotron design is *so to configure the system that we will bring this about.* Then the gyrotron becomes a simple device of predictable behaviour. For once that is established, the equations separate into independent modal equations each with its own dispersion relation just as for empty waveguide TE and TM modes. Now, finally, we want to arrange that the dispersion relations for all but (ideally) one mode will, at our chosen excitation frequency ω_e indicate a 'cut-off' or at least non-amplifying condition, and so we recover the concept of the gyrotron as a *single-mode device.*

Within the limitations of our assumptions—velocity distributions narrow in $v_{\perp 1}$ and v_{z1}, uniform in ϕ_1, and operation at the fundamental cyclotronic harmonic—our analysis gives a very flexible approach to the investigation of alternative gyro-

tron configurations, whose behaviour can now be analysed simply by evaluation of the *coupling integral coefficients* $\mathrm{EE}_{\psi n}^{\xi m}$, $\mathrm{EH}_{\psi n}^{\xi m}$ without reworking the entire theory.

We close with an example of a dispersion equation. Consider a TE mode and suppose all integrals coupling it to other modes are zero. Then the q_n equation drops out and we have just the two V_n^{TE}, I_n^{TE} equations. There is no $\int e_z J_z \, d\Sigma$ coupling integral, so a bit of algebra gives:

$$\left[\frac{1}{\mu_0}\left(\frac{\omega^2}{c^2} - k^2 - (k_{cn})^2\right) + j\omega(\omega - kv_{z1})\right.$$
$$\times \left\{\frac{-2\pi j}{(\omega - kv_{z1})^2 - \omega_{ce1}^2}\frac{e}{m_0}\right.$$
$$\times \left[\mathrm{EE}_x^{x(\mathrm{f})} + \mathrm{EE}_y^{y(\mathrm{f})} - \tfrac{1}{2}(\mathrm{EE}_x^{x(\mathrm{rel})} + \mathrm{EE}_y^{y(\mathrm{rel})}) - v_{z1}(\mathrm{EH}_y^x - \mathrm{EH}_x^y)\frac{k}{\mu_0\omega}\right]$$
$$\left.\left. + \pi j \frac{\omega_{ce1}^2}{((\omega - kv_{z1})^2 - \omega_{ce1}^2)^2}\frac{e}{m_0}(\mathrm{EE}_x^{x(\mathrm{rel})} + \mathrm{EE}_y^{y(\mathrm{rel})})\right\}\right] V_n$$
$$= \frac{k}{\mu_0\omega} g_V + g_I - \frac{(\omega - kv_{z1})2\pi j}{(\omega - kv_{z1})^2 - \omega_{ce1}^2}\frac{e}{m_0} v_{z1}(\mathrm{EH}_y^x - \mathrm{EH}_x^y)\frac{g_V}{\mu_0\omega^2} \tag{7.1}$$

putting the former RHSs as just g_I and g_V, and EE_{yn}^{xn} as EE_y^x etc., and using the fact that $\mathrm{EE}_y^x = \mathrm{EE}_x^y$, and also that $\mathrm{EH}_x^x = -\mathrm{EH}_y^y$ since $\mathbf{h}_{\perp n} = \hat{\mathbf{z}} \times \mathbf{e}_{\perp n} = -e_{yn}\hat{\mathbf{x}} + e_{xn}\hat{\mathbf{y}}$ ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ being the unit vectors).

By the analysis of Briggs (1964) and Bers (1975), we see that the actual dispersion equation is the denominator of the expression for V_n in reduced form (polynomial over polynomial) less the $(\omega \pm \omega_e)$ driving term factor. They show that the solution will, asymptotically in time (i.e. the steady-state solution), show $\exp(-j\omega t + jkz)$ dependence where $k = k_u(\omega)$ is a solution to this dispersion equation at $\omega = \omega_e$ (assuming that the dispersion equation reveals no absolute instabilities). The equation here is, then:

$$\left(\frac{\omega^2}{c^2} - k^2 - k_{cn}^2\right) = \frac{-2\pi\mu_0\omega(\omega - kv_{z1})\frac{e}{m_0}\left[\mathrm{EE}^{(\mathrm{f})} - \tfrac{1}{2}\mathrm{EE}^{(\mathrm{rel})} - \frac{v_{z1}k}{\mu_0\omega}\mathrm{EE}^{(\mathrm{f})}\right]}{[(\omega - kv_{z1})^2 - \omega_{ce1}^2]}$$
$$+ \frac{\pi\mu_0\omega(\omega - kv_{z1})\omega_{ce1}^2(e/m_0)\mathrm{EE}^{(\mathrm{rel})}}{[(\omega - kv_{z1})^2 - \omega_{ce1}^2]^2} \tag{7.2}$$

writing $\mathrm{EE}^{(\mathrm{f})} = \mathrm{EE}_x^{x(\mathrm{f})} + \mathrm{EE}_y^{y(\mathrm{f})}$, $\mathrm{EE}^{(\mathrm{rel})} = \mathrm{EE}_x^{x(\mathrm{rel})} + \mathrm{EE}_y^{y(\mathrm{rel})}$, and recalling that $\mathrm{EH}_y^x - \mathrm{EH}_x^y = \mathrm{EE}_x^{x(\mathrm{f})} + \mathrm{EE}_y^{y(\mathrm{f})}$. This term—appearing as the $(v_{z1}k/\mu_0\omega)\mathrm{EE}^{(\mathrm{f})}$ term under this switch—represents the 'Weibel' instability due to $\mathbf{v} \times \mathbf{B}$ interaction. The $\mathrm{EE}^{(\mathrm{rel})}$ terms (particularly the second) represent the relativistic effects: their origin is easily traced to the $\gamma^{(1)}$ and $d\gamma^{(1)}/d\tau$ parts of the original equation.

Writing the equation as:

$$\left(\frac{\omega^2}{c^2} - k^2 - k_{cn}^2\right)[(\omega - kv_{z1})^2 - \omega_{ce1}^2]^2 = -2\pi\mu_0\omega(\omega - kv_{z1})\frac{e}{m_0}$$
$$\times \left[[(\omega - kv_{z1})^2 - \omega_{ce1}^2]\left[\mathrm{EE}^{(\mathrm{f})} - \tfrac{1}{2}\mathrm{EE}^{(\mathrm{rel})} - \frac{v_{z1}k}{\mu_0\omega}\mathrm{EE}^{(\mathrm{f})}\right] - \omega_{ce1}^2\tfrac{1}{2}\mathrm{EE}^{(\mathrm{rel})}\right]$$

reveals the uncoupled waveguide and cyclotron modal form of the LHS.

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