CHAPTER V

5.0   CYCLOTRON MODES

5.1   Introduction

As something of a preface to the next chapter, I would like to extend and clarify the concept of cyclotron modes. This little chapter does just that. It also serves as a simple application of some of the ideas in Chapters II and IV.

Again, I will here restrict consideration to the non-relativistic or 'classical' case.

5.2   Definition of the modes

5.2.1.   To define cyclotron modes, we presume a system with an external B_o field along $\hat{z}$ much as in Chapter II, and suppose there is no r.f. $\underline{E}$ or $\underline{B}$ field: this assumption of the absence of an $\underline{E}$ field I will call the *drift space assumption*. So the Lorentz equation is just

$$\frac{d}{d\tau} \underline{v} = -\frac{e}{m} \underline{v} \times \underline{B}_o$$

and, defining $\omega_c := (e/m)B_o$, the $\omega_{ceo} : \omega_{cel}$ distinction being unnecessary here, we can express our equations as:

$$\frac{d}{d\tau} x = v_x \qquad \frac{d}{d\tau} v_x = -\omega_c v_y \qquad \frac{d}{d\tau} v_z = 0$$

$$\frac{d}{d\tau} y = v_y \qquad \frac{d}{d\tau} v_y = \omega_c v_x \qquad \frac{d}{d\tau} z = v_z = v_{zl}$$

(1)

and these will henceforth be taken as the fundamental equations of cyclotron mode systems.

Now, in the transverse velocity equations, we see that the operator $\frac{1}{\omega_c} \frac{d}{d\tau}$ gives a *natural complex structure* to the transverse

velocity space in the sense outlined at the end of Chapter IV.

For, putting a transverse velocity vector as $\underline{v} = v_x \hat{\underline{x}} + v_y \hat{\underline{y}}$, and

defining $J := \dfrac{1}{\omega_c} \dfrac{d}{d\tau}$, we find that $J^2 \underline{v} = - \underline{v}$, since

$$\frac{1}{\omega_c} \frac{d}{d\tau} \frac{1}{\omega_c} \frac{d}{d\tau} (v_x \hat{\underline{x}} + v_y \hat{\underline{y}}) = \frac{1}{\omega_c} \frac{d}{d\tau} (-v_y \hat{\underline{x}} + v_x \hat{\underline{y}}) = - (v_x \hat{\underline{x}} + v_y \hat{\underline{y}})$$

i.e. $\dfrac{1}{\omega_c} \dfrac{d}{d\tau} \dfrac{1}{\omega_c} \dfrac{d}{d\tau} \underline{v} = - \underline{v}.$

This in turn suggests a complexification in transverse velocity

space defined by:

$$v_{\pm} := v_x \pm j v_y . \tag{2}$$

Working with this *simple complexification*, we find we can diagonalise

the velocity equations thus: multiply the second $(\dfrac{d}{d\tau} v_y)$ equation

by j and then add and subtract:

$$\frac{d}{d\tau} (v_x + j v_y) = j\omega_c (v_x + j v_y)$$

$$\frac{d}{d\tau} (v_x - j v_y) = - j\omega_c (v_x - j v_y)$$

so $$\frac{d}{d\tau} v_{\pm} = \pm j\omega_c v_{\pm} . \tag{3.1}$$

It is easy to see that we can extend this to a diagonalisation

of all the transverse equations, for

$$\frac{d}{d\tau} x = v_x = \frac{1}{\omega_c} \frac{d}{d\tau} v_y$$

$$\frac{d}{d\tau} y = v_y = - \frac{1}{\omega_c} \frac{d}{d\tau} v_x$$

so $$\frac{d}{d\tau} (\omega_c x - v_y) = 0$$

$$\frac{d}{d\tau} (\omega_c y + v_x) = 0 .$$

Define $r_{\pm} := x \pm jy$ and perform analogous manipulation to the above to get:

$$\frac{d}{d\tau} (j\omega_c r_+ - v_+) = 0$$

$$\frac{d}{d\tau} (j\omega_c r_- + v_-) = 0 .$$

So, defining $\sigma_{\pm} = v_{\pm} \mp j\omega_c r_{\pm}$, which notation I choose expressly so as *NOT* to conflict with established notations here, we obtain

$$\frac{d}{d\tau} \sigma_{\pm} = 0 \tag{3.2}$$

which, together with:

$$\frac{d}{d\tau} v_{\pm} = \pm j\omega_c v_{\pm}$$

gives the four unlinked or diagonalised equations.

5.2.2   Now let's look at quite a different derivation of these equations.   Let us try to diagonalise the matrix appearing in:

$$\frac{d}{d\tau} \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega_c \\ 0 & 0 & \omega_c & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} \tag{4}$$

by finding the eigenvectors of this matrix.   Putting the matrix as $\underline{A}$, the vectors as $\underline{p}$, these must solve:

$$(\underline{A} - \lambda)\underline{p} = 0$$

which will have a solution if $\det(\underline{A} - \lambda\underline{I}) = 0$.   That determinant is

$$\lambda^2 (\lambda^2 + \omega_c^2)$$

giving solutions

$$\lambda = 0$$

$$\lambda = \pm j\omega_c \ ,$$

as eigenvalues, so giving the *eigenvector* equations:

1. $\underline{A} \ \underline{p} = 0$ (5.1)

2. $\underline{A} \ \underline{p} = \pm j\omega_c \ \underline{p}$ (5.2)

Now $\underline{A} \ \underline{p}$, written as a row vector, is $[v_x \ \ v_y \ \ -\omega_c v_y \ \ \omega_c v_x]$ so these equations give respectively:

1. $v_x = v_y = 0$ (6.1)

2. $v_x = \pm j\omega_c x$ $-\omega_c v_y = \pm j\omega_c v_x$

       $v_y = \pm j\omega_c y$ $\omega_c v_x = \pm j\omega_c v_y$ (6.2)

as the equations of the two groups of eigenvector. Observe that the $v_x, \ v_y$ equations in (2.) reduce to just

$$v_x = \pm jv_y \ .$$

NOW PATENTLY THESE DO NOT HAVE A SOLUTION FOR $x, \ y, \ v_x, \ v_y$ *REAL. BECAUSE THE EIGENVALUES ARE IMAGINARY, WE DO NOT HAVE REAL EIGENVECTORS.*

So the matrix does *NOT* admit of a real diagonalization. Instead, we proceed to make the *standard complexification* of x, y, v_x and v_y, which, incidentally, merely corresponds for any vector space to the replacement of its real scalars by complex ones.

In the standard complexification, the eigenvectors are now given by any $[x \ \ y \ \ v_x \ \ v_y]$ vector of the following forms:

1. $[\alpha_x \quad \alpha_y \quad 0 \quad 0]$ $[\beta_x \quad \beta_y \quad 0 \quad 0]$

2. $[\frac{1}{\omega_c}\gamma \quad -\frac{j}{\omega_c}\gamma \quad j\gamma \quad \gamma]$ $[\frac{1}{\omega_c}\delta \quad \frac{j}{\omega_c}\delta \quad -j\delta \quad \delta]$

to be consistent with the equations above.   Note that we can choose ANY two linearly independent vectors with their v_x and v_y components zero for (1.), but in (2.) the direction of the eigenvectors is completely prescribed, and γ and δ represent their arbitrary magnitudes. Now this appears to bear absolutely no relation to the $v_{\pm}$, $\sigma_{\pm}$ modes; but put an arbitrary $[x \quad y \quad v_x \quad v_y]$ in terms of our eigenvectors, which must form a *basis* for the $[x \quad y \quad v_x \quad v_y]$ vector space:

$$\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \sigma_+ \begin{bmatrix} \alpha_x \\ \alpha_y \\ 0 \\ 0 \end{bmatrix} + \sigma_- \begin{bmatrix} \beta_x \\ \beta_y \\ 0 \\ 0 \end{bmatrix} + v_+ \begin{bmatrix} \frac{1}{\omega_c}\gamma \\ -\frac{j}{\omega_c}\gamma \\ j\gamma \\ \gamma \end{bmatrix} + v_- \begin{bmatrix} \frac{1}{\omega_c}\delta \\ \frac{j}{\omega_c}\delta \\ -j\delta \\ \delta \end{bmatrix} \qquad (7)$$

so that $x = \sigma_+\alpha_x + \sigma_-\beta_x + \frac{1}{\omega_c}(v_+\gamma + v_-\delta)$

$y = \sigma_+\alpha_y + \sigma_-\beta_y - \frac{j}{\omega_c}(v_+\gamma - v_-\delta)$

$v_x = j(v_+\gamma - v_-\delta)$

$v_y = (v_+\gamma + v_-\delta)$

and solve for σ_+, σ_-, v_+, v_-;

the last two equations give:

$v_x + jv_y = 2jv_+\gamma$

$v_x - jv_y = -2jv_-\delta$

and these in the first two give:

$$x = \sigma_+ \alpha_x + \sigma_- \beta_x + v_y/\omega_c$$

$$y = \sigma_+ \alpha_y + \sigma_- \beta_y - v_x/\omega_c$$

or

$$x \pm jy = \sigma_+(\alpha_x \pm j\alpha_y) + \sigma_-(\beta_x \pm j\beta_y) + \frac{1}{\omega_c}(v_y \mp jv_x)$$

hence

$$j\omega_c r_{\pm} = \sigma_+ \alpha_{\pm} + \sigma_- \beta_{\pm} \pm v_{\pm}$$

defining $\alpha_{\pm} = j\omega_c(\alpha_x \pm j\alpha_y)$, $\beta_{\pm} = j\omega_c(\beta_x \pm j\beta_y)$.

We recover our original definitions by the choice:

$$\gamma = -\frac{j}{2} \qquad \delta = \frac{j}{2}$$

$$\alpha_+ = -1 \qquad \alpha_- = 0 \tag{8}$$

$$\beta_+ = 0 \qquad \beta_- = 1.$$

So these *modal variables* $v_{\pm}$, $\sigma_{\pm}$ correspond in fact to the coefficients in the expression of an arbitrary $[x \quad y \quad v_x \quad v_y]$ in terms of a particular choice of eigenvectors of the system matrix.

But this only works if x, y, v_x, v_y are *COMPLEX* - i.e. if we work in a *complex vector space* in the *standard complexification*.

Now we originally introduced $v_{\pm}$, $r_{\pm}$ and $\sigma_{\pm}$ using the *simple complexification* from x, y, v_x, v_y real. What this, in effect, does is this: the complex vector space just defined above is effectively an eight-dimensional real vector space. Choosing x, y, v_x, v_y *real* restricts us to a four-dimensional subspace of this. In this subspace, v_+ and v_-, and r_+ and r_- are no longer independent: in particular $v_+ = 0$ implies $v_x = v_y = 0$ and so $v_- = 0$. Furthermore, the eigenvectors themselves do *NOT* belong to this subspace.

So we cannot in this subspace find a vector with the σ_+, σ_- and v_- components all zero but a non-zero v_+ for example. However, the equations

$$\frac{d}{d\tau} \sigma_{\pm} = 0 \qquad \frac{d}{d\tau} v_{\pm} = \pm j\omega_c v_{\pm}$$

still hold for elements of this subspace just as they do throughout the complex vector space.

Let's look at this in a little more detail: we restrict our attention to the transverse section of phase space co-ordinated by x, y, v_x, v_y in a given t-plane (i.e. we'll 'hold' the t variation for the moment, since we will not 'complexify' t). Call this $N_\perp(t)$. We perform the *standard complexification* of this space, replacing its real co-ordinates with complex ones to obtain a new *complex vector space* which we can call $C N_\perp(t)$ (in the notation of Nickerson, Spencer and Steenrod [1]). So we now treat x, y, v_x, v_y as being *complex* variables: clearly this four-dimensional *complex* space corresponds to an 8-dimensional *real* one parametrised by x_r, x_i, y_r, y_i, v_{xr}, v_{xi}, v_{yr}, v_{yi}.

The *simple complexification* takes us into a four-dimensional subspace of $C N_\perp(t)$, defined in terms of its real counterpart by $x_i = y_i = v_{xi} = v_{yi} = 0$, or in terms of the complex parameters by $x = \bar{x}$, $y = \bar{y}$, $v_x = \bar{v}_x$, $v_y = \bar{v}_y$, using the bar for complex conjugate. Since the subspace is in $1:1$ correspondence with $N_\perp(t)$ itself we can refer to it AS $N_\perp(t)$.

Over the whole of $C N_\perp(t)$ we can define the variables v_+, v_-, r_+ and r_-: these are complex variables and given an alternative co-ordinate system for $C N_\perp(t)$ to x, y, v_x, v_y. We can furthermore choose σ_+ and σ_- rather than r_+ and r_-, and then $v_- = \sigma_+ = \sigma_- = 0$

defines the v_+ eigenvector line in our space.

On the subspace $N_\perp(t)$, the defining restriction above appears in terms of *THESE* variables as $v_- = \bar{v}_+$, $r_- = \bar{r}_+$, and by a quick calculation, we see also that $\sigma_- = \bar{\sigma}_+$ there too. *THIS SPACE DOES NOT CONTAIN THE EIGENVECTORS, THEN,* since if v_+ is zero, so is v_- etc.

Let's append to $C\,N_\perp(t)$ and therefore to $N_\perp(t)$ the parameter t which remains *real*, but now gives a 4 complex × 1 real dimensional space corresponding to a 9 dimensional real space.

Now trajectories are well-defined in this space, by the equations for the four complex variables, which, assuming τ is uniform with t and using $\underline{v_+}$, $\underline{\sigma_+}$ parametrisation, take the form:

$$\frac{d}{d\tau}\sigma_+ = 0 \qquad \frac{d}{d\tau}\sigma_- = 0 \qquad \frac{d}{d\tau}v_+ = j\omega_c v_+ \qquad \frac{d}{d\tau}v_- = -j\omega_c v_-$$

and $\dfrac{dt}{d\tau} = 1$, for the real t.

These equations are equivalent to 8 (9 including $\dfrac{dt}{d\tau} = 1$) equations for the parameters of the real space associated with this. So they uniquely define a trajectory all right!

Calling this 'space with t' just $CN_\perp$, so that $C\,N_\perp$ is parametrised by *complex* x, y, v_x, v_y or $\underline{v_+}$, $\underline{\sigma_+}$, and *real* t, and the $x = \bar{x}, \ldots v_y = \bar{v}_y$ subspace just $N_\perp$, let us now suppose we have a *datum* point in $N_\perp$.

Now

$$\frac{d}{d\tau}\bar{\sigma}_+ = 0 \quad \text{if} \quad \frac{d}{d\tau}\sigma_+ = 0$$

and

$$\frac{d}{d\tau}\bar{v}_+ = -j\omega_c \bar{v}_+ \quad \text{if} \quad \frac{d}{d\tau}v_+ = j\omega_c v_+ \;.$$

On $N_\perp$, $\bar{v}_+ = v_-$, so the effect is that any such trajectory *STAYS IN* $N_\perp$ *THROUGHOUT*. That is, even though *WITHIN* $N_\perp$ a trajectory is uniquely defined by the *PAIR* of equations:

$$\frac{d}{d\tau}\sigma_+ = 0 \qquad \frac{d}{d\tau}v_+ = j\omega_c v_+$$

since v_+ and v_- and σ_+ and σ_- are *DEPENDENT* there, the other equations remain consistent with these at all times. So that is why we obtain the same equations from the *simple complexification* on just the $N_\perp$ subspace.

Another, more explicit, way of looking at this is to consider the real form of the space: since the imaginary parts obey the same equations as the real parts, by linearity, the only solution given the datum (initial) conditions $x_i = y_i = v_{xi} = v_{yi} = 0$ at $\tau = 0$ is the trivial one $x_i(\tau) = y_i(\tau) = v_{xi}(\tau) = v_{yi}(\tau) = 0$ for all τ, so again the trajectory stays in $N_\perp$.

5.2.3 But when we talk of cyclotron modes, we normally mean the eigenvector solutions, which correspond to the x_r, y_r, v_{xr}, v_{yr} parts of solutions obeying our trajectory equations with

$$\sigma_+ = \sigma_- = v_- = 0 \qquad v_+ \text{ mode}$$
$$\sigma_+ = \sigma_- = v_+ = 0 \qquad v_- \text{ mode}$$
$$\sigma_+ = v_+ = v_- = 0 \qquad \sigma_- \text{ mode} \tag{9}$$
$$\sigma_- = v_+ = v_- = 0 \qquad \sigma_+ \text{ mode}$$

where, by the diagonalisation, these can maintain all along a trajectory, not just coincidentally at a point of it. So for a v_+ mode, the entire behaviour is determined by the *SINGLE* equation

$$\frac{d}{d\tau} v_+ = j\omega_c v_+$$

giving *complex* v_+ as a function of τ (or in effect, t).

NOTE THAT THESE <u>MODAL VARIABLES</u> AND THE MODES THEY CORRESPOND TO HAVE BEEN FULLY DEFINED WITHOUT INTRODUCING FILAMENTARY BEAMS, OR $e^{j\omega t}$ VARIATION, OR z,t PARAMETRISATION, JUST FROM THE DIAGONALISATION OF THE MATRIX. So now let's look at these separated modes in more detail.

From the eigenvector expansion, now, we can read off directly for, for example, the v_+ mode:

$$x = v_+ (\frac{-j}{2\omega_c}) \qquad\qquad y = v_+ (\frac{-1}{2\omega_c})$$

$$v_x = v_+/2 \qquad\qquad v_y = - v_+ \, j/_2$$

remembering $\gamma = - j/2$ here.

We also know that ALONG ANY TRAJECTORY in phase space, v_+ must obey

$$\frac{d}{d\tau} v_+ = j\omega_c v_+$$

so that $\qquad v_+ = \overset{\backsim}{v}_+ \, e^{j\omega_c \tau}$

where I use the left tilde ($\backsim$, not $\sim$) to indicate a *phasor* independent of τ (but, as in Chapter II, it can depend on up to 6 constants of the motion, some of which may now be complex).

We see that, as functions of τ (i.e. all other phase space parameters constant - we stay with one electron, one trajectory) v_x and v_y show a 90° sinusoidal phase shift:

$$v_x = \overset{\backsim}{v}_+/2 \, (\cos \omega_c \tau + j \sin \omega_c \tau)$$

$$v_y = \overset{\backsim}{v}_+/2 \, (\sin \omega_c \tau - j \cos \omega_c \tau)$$

so that if we always take $Re(v_x)$, $Re(v_y)$ as our physical solutions, we see that v_x leads v_y by $90°$ wrt $\omega_c\tau$.

Much the same pattern of phase shifts occurs between x and y, and between x and v_x, y and v_y: note that x is in phase with v_y.

Now v_+ and v_- correspond to *pure cyclotron modes*. I now introduce for the first time the term *synchronous modes* to describe the modes where $v_+ = v_- = 0$.

The behaviour of these synchronous modes is quite different. Consider σ_+: now $\alpha_+ = -1$, $\alpha_- = 0$ gives:

$$\alpha_x = j\alpha_y \quad \therefore \quad 2j\alpha_y = -1 \quad \therefore \quad \alpha_y = -\frac{1}{2j}$$

$$\alpha_x = -\frac{1}{2}$$

so we have

$$x = \sigma_+\left(-\frac{1}{2}\right)$$

$$y = \sigma_+\left(-\frac{1}{2j}\right) = \sigma_+ \cdot \frac{j}{2}$$

$$v_x = v_y = 0 ,$$

where σ_+ is *CONSTANT* along any trajectory. So now the physical solutions x and y in fact correspond to the real and imaginary parts of σ_+ and are themselves *CONSTANT* wrt τ.

Note how, then, any distribution in phase space will just be 'carried downstream' unaltered by these 'modes': i.e. a distribution of electrons in an xy-plane at some initial $z = z_1$ will re-appear quite unaltered at some subsequent $z = z_2$ to which it is carried by the trajectories. This property maintains in a weaker form for the cyclotron modes too, there now being just a rotation at frequency ω_c imposed on the downstream drift. This is crucial to understanding the physics of cyclotron mode systems: the modes are not *dynamic*

but *distributional*. This will be more clear if we consider a specific example.

All treatments of cyclotron modes elsewhere consider *ONLY* the special case of a *filamentary beam*, further restricted to $e^{j\omega t}$ *variation*. Note that these *TWO* restrictions are not concomitant to each other, nor, as we have seen, essential to the concept of cyclotron (synchronous) modes which can be fully developed without them.

First, the filamentary beam assumption: as shown in Chapter II, a filamentary beam is conveniently parametrised by z and t if we assume v_z constant. v_z is certainly a *weak constant* or constant of the motion as defined in Chapter II, as we see directly from the opening equations of the present chapter. We also assume it to be a *strong constant* over the filamentary beam manifold, which just means we assume it to be the same for successive electrons. Let us, as before, write it as v_{z1}.

The $\frac{d}{d\tau}$ for the filamentary beam takes the form $\partial_t + v_{z1}\partial_z$ then, and so our modal equations become:

$$(\partial_t + v_{z1}\partial_z)v_{\underline{+}} \mp j\omega_c v_{\underline{+}} = 0 \tag{10.1}$$

$$(\partial_t + v_{z1}\partial_z)\sigma_{\underline{+}} = 0 . \tag{10.2}$$

Now for the second assumption: that of $e^{j\omega t}$ dependence. This reduces us to *(real) normal modes* in the sense of Chapter IV. The choice of $e^{j\omega t}$ means that we choose now to look at filamentary beams which show sinusoidal variation in time at any fixed z-plane. It in fact turns out that the separated modes all show a pure *rotation* with t in any fixed z-plane.

Now expressing the variables in terms of z-dependent *phasors* thus:

$$v_+(z,t) = \tilde{v}_+(z) e^{j\omega t}$$

$$\sigma_+(z,t) = \tilde{\sigma}_+(z) e^{j\omega t}$$

where I use the right tilde (~, not ◡) to distinguish this $e^{j\omega t}$ phasor from the earlier $e^{j\omega_c \tau}$ ones, we obtain equations for $\tilde{v}_+$, $\tilde{\sigma}_+$ as:

$$\partial_z \tilde{v}_+ + j\left(\frac{\omega}{v_{zl}} \mp \frac{\omega_c}{v_{zl}}\right)\tilde{v}_+ = 0 \qquad (11.1)$$

$$\partial_z \tilde{\sigma}_+ + j\frac{\omega}{v_{zl}} \tilde{\sigma}_+ = 0 \qquad (11.2)$$

and defining $k_e := \dfrac{\omega}{v_{zl}}$, $k_c = \dfrac{\omega_c}{v_{zl}}$, and solving these, we find:

$$\tilde{v}_+(z) = \tilde{K}_{v+} e^{-j(k_e \mp k_c)z} \qquad (12.1)$$

$$\tilde{\sigma}_+(z) = \tilde{K}_{\sigma+} e^{-jk_e z} \qquad (12.2)$$

where the phasors $\tilde{K}_{v+}$, $\tilde{K}_{\sigma+}$ are independent of z,t and vary according to the initial conditions only. In the book by Louisell [2] these appear as $|c_+|e^{j\theta_+}$ and $|b_+|e^{j\phi_+}$ respectively to separate explicitly their modulus and argument.

We now put these into the equations given earlier to abstract the x, y, v_x, v_y dependence for the separated modes. For example, for the v_+ mode:

$$x = -\frac{j}{2\omega_c} \tilde{K}_{v+} e^{-j(k_e - k_c)z} e^{j\omega t} \qquad (13.1)$$

$$y = -\frac{1}{2\omega_c} \tilde{K}_{v+} e^{-j(k_e - k_c)z} e^{j\omega t}$$

$$v_x = \frac{1}{2} \tilde{K}_{v+} e^{-j(k_e - k_c)z} e^{j\omega t}$$

$$v_y = -\frac{j}{2} \tilde{K}_{v+} e^{-j(k_e - k_c)z} e^{j\omega t} \qquad (13.4)$$

or for the σ_+:

$$x = -\frac{1}{2}\tilde{K}_{\sigma+}\, e^{-jk_e z}\, e^{j\omega t} \qquad\qquad v_x = 0 \qquad (13.5)$$

$$y = \frac{j}{2}\tilde{K}_{\sigma+}\, e^{-jk_e z}\, e^{j\omega t} \qquad\qquad v_y = 0 . \qquad (13.6)$$

We can now examine the actual 'appearance' of the resulting modes, and this is excellently considered in Louisell [2], which considerably expands the treatment given by Siegman [3]. In particular, the *distributional* character of the modes comes out clearly: that is, the $e^{j\omega t}$ variation is actually imposed by the manner of 'injection' of electrons, or their distribution at some $z = z_1$ entry plane as a function of time rather than by *dynamical* processes as in an acoustic or 'space charge' plasma wave. To put it another way, each electron simply goes about its business quite unaware that it is constituting part of a wave pattern: there are no forces BETWEEN electrons WITHIN the wave: the trajectories are determined entirely by the entry conditions for each electron. I have previously referred to this as the 'carrying downstream' property. It works somewhat differently for *synchronous* and for *cyclotron* modes.

For synchronous modes, each individual electron travels in a straight line in the z direction: $v_x = v_y = 0$. To produce an $e^{j\omega t}$ rotation at every z-plane we require a distribution of electrons which AT FIXED t is *helical*. As t varies, this entire helical distribution moves bodily downstream at $v_z = v_{z1}$ WITHOUT ROTATING, so the pitch of the helix DIRECTLY produces the 'rotation' at any fixed z-plane, and indeed is related to the resulting apparent ω by:

$$p = \mp\frac{2\pi v_{z1}}{\omega} .$$

The σ_+ and σ_- modes differ simply in that the helix is wound left-hand in one and right-hand in the other.   Diagram 1 shows the left-hand or σ_+ version: this is Louisell's fig. 2.8 [2].

The mechanism is a little more subtle for the cyclotron modes, for now each individual electron is ROTATING AT FREQUENCY ω_c AS WELL AS travelling in the z direction at $v_z = v_{z1}$.   To obtain an $e^{j\omega t}$ rotation of the point of intersection of the beam at every z-plane NOW, we again have a distribution of electrons which is helical at any fixed t, BUT NOW THIS ENTIRE DISTRIBUTION MOVES BODILY DOWNSTREAM AT $v_z = v_{z1}$ AND AT THE SAME TIME ROTATES WITH FREQUENCY ω_c.

This has the effect that the cross-over point at which the pitch changes from left-hand to right-hand is shifted from zero to ω_c and we have:

$$p = \mp \frac{2\pi v_{z1}}{(\omega \mp \omega_c)} = - \frac{2\pi v_{z1}}{(\underset{-}{+}\omega - \omega_c)} .$$

This means that for the v_- mode the pitch is necessarily right-handed, but for the v_+ mode it is left-handed if ω is to be greater than ω_c, but again right-handed to effect an ω less than ω_c.   Indeed, at $\omega = \omega_c$ the pitch is infinite for the v_+ mode and the electrons lie along a straight line (at any fixed t):   the 'point of injection' itself now rotates at ω_c.   Diagram 2, for the v_+ mode, is Louisell's fig. 2.6 [2].

I now introduce the terms for the individual modes:

the v_+ mode we call the *fast cyclotron mode*

the v_- the *slow cyclotron mode*

the σ_+ mode we call the *left-hand synchronous mode*

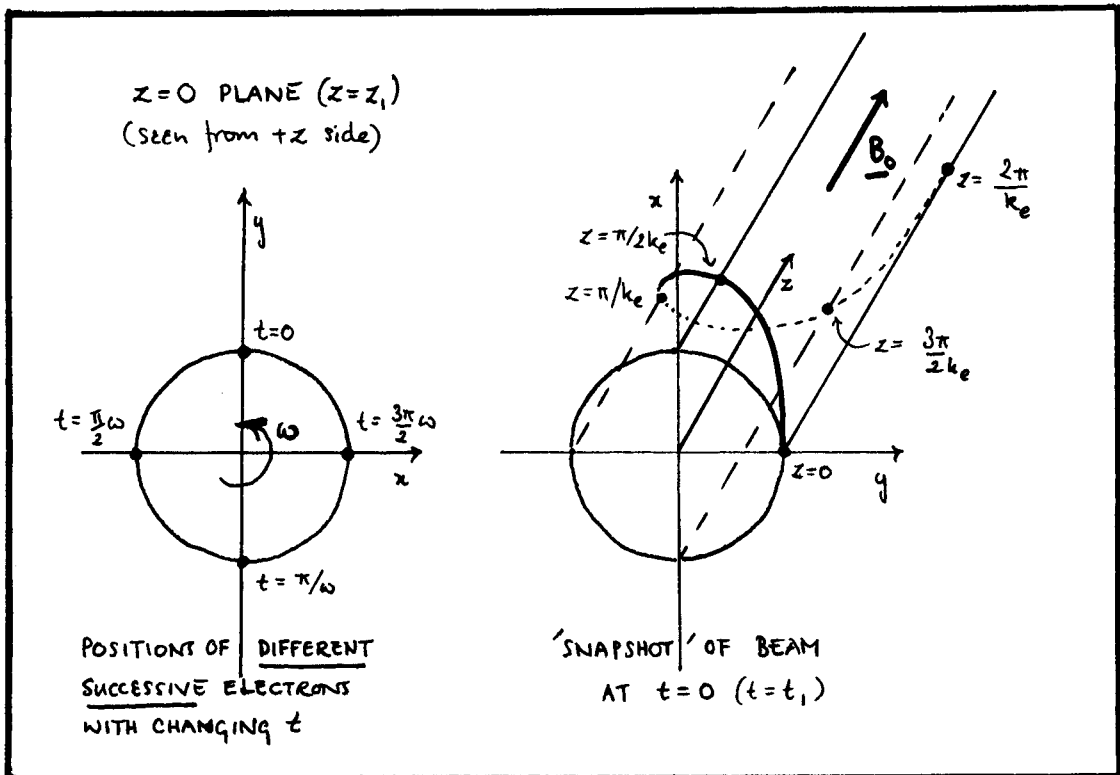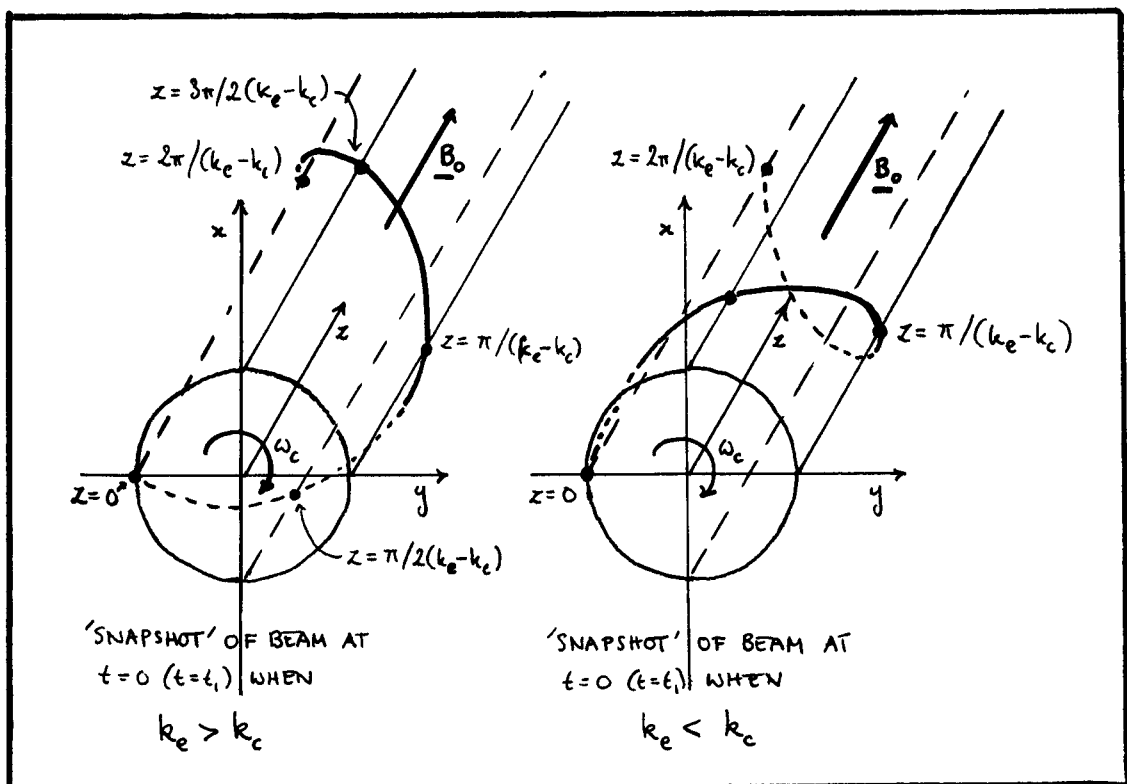the σ_- the *right-hand synchronous mode*

DIAGRAM 1

z=0 PLANE $(z=z_1)$
(seen from +z side)

$t=0$

$t=\frac{\pi}{2}\omega$ ω $t=\frac{3\pi}{2}\omega$

$t=\pi/\omega$

POSITIONS OF **DIFFERENT**
SUCCESSIVE ELECTRONS
WITH CHANGING t

B_0

$z=\frac{2\pi}{k_e}$

$z=\pi/2k_e$

$z=\pi/k_e$

$z=\frac{3\pi}{2k_e}$

$z=0$

'SNAPSHOT' OF BEAM
AT $t=0$ $(t=t_1)$

DIAGRAM 2

$z=3\pi/2(k_e-k_c)$

$z=2\pi/(k_e-k_c)$

B_0

$z=\pi/(k_e-k_c)$

ω_c

$z=0$

$z=\pi/2(k_e-k_c)$

'SNAPSHOT' OF BEAM AT
$t=0$ $(t=t_1)$ WHEN

$$k_e > k_c$$

$z=2\pi/(k_e-k_c)$

B_0

$z=\pi/(k_e-k_c)$

ω_c

$z=0$

'SNAPSHOT' OF BEAM AT
$t=0$ $(t=t_1)$ WHEN

$$k_e < k_c$$

and I remark that the variables $a_{1\pm}$, $a_{2\pm}$ used by Louisell correspond
to the present $v_\pm$, $\sigma_\pm$ normalised wrt the energy of the wave.

5.3   Fourier Analysis

5.3.1   Now it is my opinion that these singular solutions are not
really so very important.   A much more comprehensive picture of what
is happening can be obtained by looking at the general solutions
given by a Fourier analysis.   I did this in outline in Chapter IV
and now repeat this analysis in more detail.

Again we consider a filamentary beam, parametrized by z,t.
Remember, as shown in Chapter II, we can re-parametrize all phase
space in terms of filamentary beams by introducing five other co-ordinates
which are constant along trajectories; so this model is not actually
restrictive.

Now I will make one significant departure from the analysis
in Chapter IV.   In cyclotron mode systems we are generally more
concerned with known conditions at some *entry plane*, which I will set
as z = 0.   This suggests that our initial conditions or *datum space*
is not actually for t = 0 now but for z = 0, and this suggests the use
of a Laplace or one-sided transform in z, and this will then pick up
the entry plane data as boundary conditions.   Furthermore we may
assume *zero-state* initial conditions in time – that is everything zero
at t = 0 (or at some arbitrary time in the remote past or t = −∞, which
enables us actually to regard the t transform as a two-sided or
Fourier one).

Our equations are:

$$(\partial_t + v_{z1}\partial_z)v_\pm \mp j\omega_c v_\pm = 0$$

$$(\partial_t + v_{z1}\partial_z)\sigma_\pm = 0$$

and we may postulate that we are using *EITHER* complexification for

$v_{\pm} := v_x \pm jv_y$. Whichever we assume, it will not affect our analysis.

The one-sided or Laplace transform by e^{-jkz} involves a

$k_+ := k_r - j\varepsilon$ for $\varepsilon > 0$ variable, as outlined at the beginning of

Chapter IV. In other words, we transform into a line in the k-plane

BELOW the real axis (just below will in fact suffice).

So $\qquad L_z \rightarrow \partial_t v_{\pm}(k,t) + v_{z1}[jkv_{\pm}(k,t) - v_{\pm}(z=0,t)] \mp j\omega_c v_{\pm}(k,t) = 0$

$$\partial_t \sigma_{\pm}(k,t) + v_{z1}[jk\sigma_{\pm}(k,t) - \sigma_{\pm}(z=0,t)] = 0$$

$$"F"_t \rightarrow -j\omega v_{\pm}(k,\omega) + v_{z1}jkv_{\pm}(k,\omega) \mp j\omega_c v_{\pm}(k,\omega) = v_{z1} \left| v_{\pm}(z=0,\tau)e^{j\omega\tau}d\tau \right.$$

$$-j\omega\sigma_{\pm}(k,\omega) + v_{z1}jk\sigma_{\pm}(k,\omega) = v_{z1} \left| \sigma_{\pm}(z=0,\tau)e^{j\omega\tau}d\tau \right.$$

or: $\qquad [k - k_e \mp k_c]v_{\pm}(k,\omega) = -j \left| v_{\pm}(z=0,\tau)e^{j\omega\tau}d\tau \right.$

$$[k - k_e]\sigma_{\pm}(k,\omega) = -j \left| \sigma_{\pm}(z=0,\tau)e^{j\omega\tau}d\tau \right.$$

where τ is just a variable of integration. From these, then:

$$v_{\pm}(z,t) = -j \left| \frac{dk}{2\pi}e^{jkz} \right| \frac{d\omega}{2\pi}e^{-j\omega t} \frac{\left| v_{\pm}(z=0,\tau)e^{j\omega\tau}d\tau \right.}{\left[k \mp k_c - \dfrac{\omega}{v_{z1}} \right]} \qquad (14.1)$$

$$\sigma_{\pm}(z,t) = -j \left| \frac{dk}{2\pi}e^{jkz} \right| \frac{d\omega}{2\pi}e^{-j\omega t} \frac{\left| \sigma_{\pm}(z=0,\tau)e^{j\omega\tau}d\tau \right.}{\left[k - \dfrac{\omega}{v_{z1}} \right]} \qquad (14.2)$$

which, apart from the switch of the initial conditions into z rather

than t, are much as in Chapter IV.

Performing the $d\omega$ integration innermost, $Im(\omega)$ must be $\geqslant 0$

along the inversion line (which may be $Re(\omega) : \omega_{io} = 0$), but $Im(k) < 0$,

indeed $Im(k) = -j\varepsilon$, so we have the situation that for $(t-\tau) < 0$

we close the contour above and enclose no poles, but for $(t - \tau) > 0$,

closing it below, we pick up the poles at $k \mp k_c$ and at k respectively.

So $\quad v_{\pm}(z,t) = - j v_{z1} \frac{2\pi j}{2\pi} \left| \frac{dk}{2\pi} e^{jkz} \right| H(t - \tau) v_{\pm}(z = 0, \tau) e^{-j(v_{z1}k \mp \omega_c)(t - \tau)} \, d\tau$

$$= \frac{v_{z1}}{2\pi} \left| H(t - \tau) v_{\pm}(z = 0, \tau) e^{+j\omega_c(t-\tau)} \right| e^{jk(z - v_{z1}(t-\tau))} \, dk \, d\tau \quad (15.1)$$

and similarly

$$\sigma_{\pm}(z,t) = \frac{v_{z1}}{2\pi} \left| H(t - \tau) \sigma_{\pm}(z = 0, \tau) \right| e^{jk(z - v_{z1}(t-\tau))} \, dk \, d\tau \, . \quad (15.2)$$

Now, here $\quad \left| \begin{matrix} \infty - j\varepsilon \\ -\infty - j\varepsilon \end{matrix} \right. e^{jk\psi} dk = e^{\varepsilon\psi} \left| \begin{matrix} \infty \\ -\infty \end{matrix} \right. e^{jk\psi} dk = e^{\varepsilon\psi} \delta(\psi) 2\pi$

but $\quad 2\pi e^{\varepsilon\psi} \delta(\psi) = 2\pi e^{\varepsilon 0} \delta(\psi) = 2\pi \delta(\psi)$, so:

$$v_{\pm}(z,t) = v_{z1} \left| H(t - \tau) v_{\pm}(z = 0, \tau) e^{+j\omega_c(t-\tau)} \delta(z - v_{z1}(t - \tau)) \, d\tau \right.$$

but $\quad \delta(z - v_{z1}(t - \tau)) = \frac{1}{|v_{z1}|} \delta(\frac{z}{v_{z1}} - (t - \tau))$, giving:

$$v_{\pm}(z,t) = H(z) \, v_{\pm}(z = 0, t - \frac{z}{v_{z1}}) \, e^{+jk_c z} \quad (16.1)$$

$$\sigma_{\pm}(z,t) = H(z) \, \sigma_{\pm}(z = 0, t - \frac{z}{v_{z1}}) \quad (16.2)$$

which graphically illustrates the 'carrying downstream' or *distributional*

nature of these modes.

5.3.2   The dispersion equation for the system as a whole is obtained

as the product of the denominators for the diagonalised system (which

will occur as such if we take the transform on the x, y, v_x, v_y

equations and eliminate all but one variable or form the determinant

of the transformed equations) and this is

$$[k - k_c - \frac{\omega}{v_{zl}}][k + k_c - \frac{\omega}{v_{zl}}][k - \frac{\omega}{v_{zl}}]^2 = 0$$

or $$[(k - \frac{\omega}{v_{zl}})^2 - k_c^2][k - \frac{\omega}{v_{zl}}]^2 = 0 \qquad (17)$$

where the square on the second factor is really merely a reminder that this corresponds to an original eigenvalue of multiplicity 2.

We can obtain this direct from the original equations by assuming an exp $(jkz - j\omega t)$ solution:

$$(\partial_t + v_{zl}\partial_z)x = v_x \qquad (\partial_t + v_{zl}\partial_z)v_x = -\omega_c v_y$$

$$(\partial_t + v_{zl}\partial_z)y = v_y \qquad (\partial_t + v_{zl}\partial_z)v_y = \omega_c v_x$$

so

$$(-j\omega + v_{zl}jk)x = v_x \qquad (-j\omega + v_{zl}jk)v_x = -\omega_c v_y$$

$$(-j\omega + v_{zl}jk)y = v_y \qquad (-j\omega + v_{zl}jk)v_y = \omega_c v_x .$$

Put the equations at left into those at right, first dividing through by v_{zl}:

$$j(k - \frac{\omega}{v_{zl}})j(k - \frac{\omega}{v_{zl}})x + jk_c(k - \frac{\omega}{v_{zl}})y = 0$$

$$j(k - \frac{\omega}{v_{zl}})j(k - \frac{\omega}{v_{zl}})y - jk_c(k - \frac{\omega}{v_{zl}})x = 0$$

which clearly has a solution for x,y if $(k - \frac{\omega}{v_{zl}}) = 0$, which is satisfied then by ANY x,y, these being our synchronous modes, or else, dividing through by $(k - \frac{\omega}{v_{zl}})$ and eliminating x from the resulting equations:

$$(k_c^2 - (k - \frac{\omega}{v_{zl}})^2)y = 0$$

which has a solution if $k_c^2 - (k - \frac{\omega}{v_{z1}})^2 = 0$, and this is the other term of our dispersion equation.

Observe that from the solutions to $v_\pm(z,t)$, $\sigma_\pm(z,t)$ we can recover $x(z,t)$, $y(z,t)$, $v_x(z,t)$ and $v_y(z,t)$ for our filamentary beam by

$$x = \sigma_+ \alpha_x + \sigma_- \beta_x + v_+ \frac{1}{\omega_c}\gamma + v_- \frac{1}{\omega_c}\delta$$

$$y = \sigma_+ \alpha_y + \sigma_- \beta_y + v_+ (-\frac{j}{\omega_c}\gamma) + v_- \frac{j}{\omega_c}\delta$$

$$v_x = j\gamma v_+ - j\delta v_-$$

$$v_y = \gamma v_+ + \delta v_-$$

which here become, for our present definition of α, β, γ, δ:

$$x = -\frac{\sigma_+}{2} + \frac{\sigma_-}{2} - \frac{j}{2\omega_c}v_+ + \frac{j}{2\omega_c}v_- \qquad (18.1)$$

$$y = j\frac{\sigma_+}{2} + j\frac{\sigma_-}{2} - \frac{v_+}{2\omega_c} - \frac{v_-}{2\omega_c}$$

$$v_x = \frac{v_+}{2} + \frac{v_-}{2}$$

$$v_y = -j\frac{v_+}{2} + j\frac{v_-}{2} \qquad (18.4)$$

which last are of course the usual formulae as in e.g. Siegman [3].

To recover our *INDIVIDUAL real normal modes* as a special case, we set up an $e^{j\omega_o t}$ driver or source at $z = 0$ - exactly analogous to the $\delta(z)e^{j\omega_o t}$ case considered by Briggs and Bers (see Chapter IV) - which now is accommodated in the entry plane datum function.

So, putting:

$$v_\pm(z = 0,\tau) = \tilde{K}_{v\pm} e^{j\omega_o \tau}$$

$$\sigma_\pm(z = 0,\tau) = \tilde{K}_{\sigma\pm} e^{j\omega_o \tau}$$

we get:

$$v_{\underline{+}}(z,t) \;=\; H(z)\,\tilde{K}_{v\underline{+}}\, e^{-j\left(\frac{\omega_o}{v_{z1}}\mp k_c\right)z}\, e^{j\omega_o t} \tag{19.1}$$

$$\sigma_{\underline{+}}(z,t) \;=\; H(z)\,\tilde{K}_{\sigma\underline{+}}\, e^{-j\frac{\omega_o}{v_{z1}}z}\, e^{j\omega_o t} \tag{19.2}$$

which are real normal modes in the strict sense defined in Chapter

IV since the k_o, ω_o of the exp $j(\omega_o t - k_o z)$ obey $D(k_o,\omega_o) = 0$ and are

both real.

Evidently these $v_{\underline{+}}(z=0,\tau)$ and $\sigma_{\underline{+}}(z=0,\tau)$ forms again

correspond to a 'rotating point of injection'.

We now see too, that if $v_{\underline{+}}(z=0,\tau)$ is in $N_{\underline{1}}$, or that

$v_-(z=0,\tau) = \bar{v}_+(z=0,\tau)$, then this holds for the entire trajectory

since:

$$\bar{v}_+(z,t) \;=\; H(z)\,\bar{v}_+(z=0,t-\frac{z}{v_{z1}})\,(\overline{e^{jk_c z}})$$

$$\;=\; H(z)\, v_-(z=0,t-\frac{z}{v_{z1}})\, e^{-jk_c z}$$

$$\;=\; v_-(z,t)$$

and the result holds trivially for σ_+ and σ_- likewise.

Clearly the Fourier transform holds for the full *standard

complexification* space, $CN_{\underline{1}}$, AND for the space of *simple complexification*

too.